A (mathematical) Introduction to Ocean Circulation Box Models

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Outline

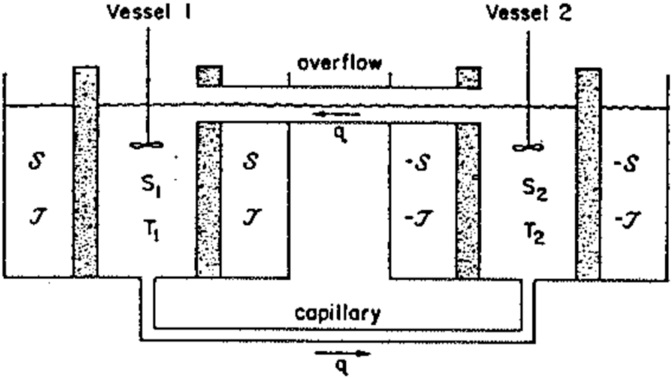
- What are Box Models?
- Stommel's Box Model
- Cessi's Model
- Roberts and Saha's Model

What are Box Models?

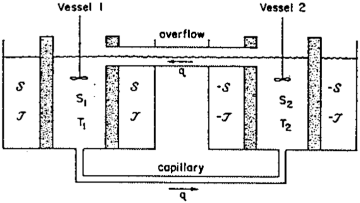
(And why do we use them?)

- What is the simplest model which can accurately describe ocean phenomena?
- A box model divides the ocean into large “boxes”, and makes the assumption that the water in each box is well mixed.
- This allows the mathematician to write down low dimensional differential equations governing the behavior of the water, which can then be studied with dynamical systems techniques.
- The point is to look at large scale, long term behavior, as opposed to detailed behavior, but can still give insight into the real system.

Stommel's Two Box Model



Stommel's Two Box Model



$kq = \rho_1 - \rho_2$

$q > 0$ if the flow goes from tank 1 to tank 2, and $q < 0$ otherwise.

$\rho = \rho_0(1 - \alpha T + \beta S)$

Stommel's Two Box Model

$$
\begin{aligned}
\frac{dT_1}{dt} &= c(\mathcal{T} - T_1) - |q|T_1 + |q|T_2 \\
\frac{dT_2}{dt} &= c(-\mathcal{T} - T_2) + |q|T_1 - |q|T_2 \\
\frac{dS_1}{dt} &= d(\mathcal{S} - S_1) - |q|S_1 + |q|S_2 \\
\frac{dS_2}{dt} &= d(-\mathcal{S} - S_2) + |q|S_1 - |q|S_2
\end{aligned}
$$

Stommel's Two Box Model

The symmetry of the system suggests we should look at solutions where $T_1 = -T_2$, and $S_1 = -S_2$.
Let $z = T_1 + T_2$.

$$\frac{dz}{dt} = c\mathcal{T} - c\mathcal{T} - c(T_1 + T_2) - |q|(T_1 + T_2) + |q|(T_1 + T_2) = -cz$$

So, the $T_1 = -T_2$ is invariant and attracting.
Doing this dimensional reduction we get

$$\begin{aligned} \frac{dT}{dt} &= c(\mathcal{T} - T) - 2|q|T \\ \frac{dS}{dt} &= d(\mathcal{S} - S) - 2|q|S \end{aligned}$$

Stommel's Two Box Model

Nondimensionalize!

Let $\tau = ct$, $\delta = \frac{d}{c}$, $y = \frac{T}{\mathcal{T}}$, and $x = \frac{S}{\mathcal{S}}$.

The system becomes:

$$\begin{aligned}\frac{dy}{d\tau} &= 1 - y - |f|y \\ \frac{dx}{d\tau} &= \delta(1 - x) - |f|x\end{aligned}$$

where f is the nondimensionalized flow:

$$\lambda f = -y + Rx, \quad R = \frac{\beta \mathcal{S}}{\alpha \mathcal{T}}$$

$$f = \frac{2q}{c}, \quad \lambda = \frac{ck}{4\rho_0 \alpha \mathcal{T}}$$

Stommel's Two Box Model

Equilibrium solutions occur at

$$\begin{aligned} x &= \frac{1}{1+\frac{|f|}{\delta}} \\ y &= \frac{1}{1+|f|} \end{aligned}$$

So, we have

$$\lambda f = -y + Rx = -\frac{1}{1+|f|} + \frac{R}{1+\frac{|f|}{\delta}} = \phi(f, R, \delta)$$

The existence of multiple equilibria depend on λ, R, and δ. For certain R and δ, it is possible to have three equilibrium solutions.

Stommel's Two Box Model

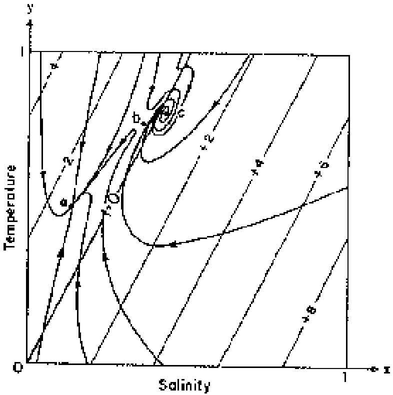


Fig. 7. The three equilibria a, b, and c for the two vessel convection experiment with $R = 2$, $\delta = 1/6$, $\lambda = 1/5$. A few sample integral curves are sketched to show the stable node *a*, the saddle b, and the stable spiral *c*.

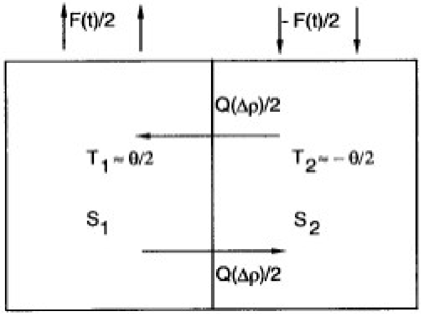
$$R = 2,\ \delta = \frac{1}{6},\ \lambda = \frac{1}{5}.$$

Cessi's Model

- Can changes in external forcing can cause a transition between the stable states found in Stommel's model?

Cessi's Model

Cessi's model is a variation on Stommel's two box model.



Cessi's Model

$$
\begin{aligned}
\rho/\rho_0 &= 1 + \alpha_S(S - S_0) - \alpha_T(T - T_0) \\
\dot{T}_1 &= -t_r^{-1}\left(T_1 - \tfrac{\theta}{2}\right) - \tfrac{1}{2}Q(\Delta\rho)(T_1 - T_2) \\
\dot{T}_2 &= -t_r^{-1}\left(T_2 + \tfrac{\theta}{2}\right) - \tfrac{1}{2}Q(\Delta\rho)(T_2 - T_1) \\
\dot{S}_1 &= \tfrac{F(t)}{2H}S_0 - \tfrac{1}{2}Q(\Delta\rho)(S_1 - S_2) \\
\dot{S}_2 &= -\tfrac{F(t)}{2H}S_0 - \tfrac{1}{2}Q(\Delta\rho)(S_2 - S_1)
\end{aligned}
$$

Cessi's Model

We again reduce the dimension of the system ($\Delta T = T_1 - T_2$, $\Delta S = S_1 - S_2$)

$$\begin{aligned}\frac{d\Delta T}{dt} &= -t_r^{-1}(\Delta T - \theta) - Q(\Delta\rho)\Delta T \\ \frac{d\Delta S}{dt} &= \frac{F(t)}{H} S_0 - Q(\Delta\rho)\Delta S\end{aligned}$$

$$Q(\Delta\rho) = \frac{1}{t_d} + \frac{q}{v}(\Delta\rho)^2$$

and nondimensionalize ($x = \frac{\Delta T}{\theta}$, $y = \frac{\alpha_S \Delta S}{\alpha_T \theta}$, $t = t_d t'$)

$$\begin{aligned}\dot{x} &= -\alpha(x-1) - x[1 + \mu^2(x-y)^2] \\ \dot{y} &= p(t) - y[1 + \mu^2(x-y)^2]\end{aligned}$$

Cessi's Model

$\alpha = t_d/t_r$ is large, so this is a fast-slow system.

The slow equation:

$$\begin{aligned} \varepsilon\dot{x} &= -(x-1) - \varepsilon x[1+\mu^2(x-y)^2] \\ \dot{y} &= p(t) - y - \mu^2 y(1-y)^2 \end{aligned}$$

The fast equation:

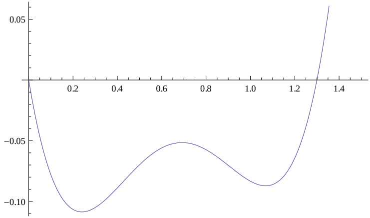
$$\begin{aligned} x' &= -(x-1) - \varepsilon x[1+\mu^2(x-y)^2] \\ y' &= \varepsilon(p(t) - y - \mu^2 y(1-y)^2) \end{aligned}$$

The (normally hyperbolic) critical manifold: $x = 1$.

Cessi's Model

For any fixed time, we can find a potential function for behavior on the critical manifold

$$U(y) = \frac{y^2}{2} + \mu^2\left(\frac{y^4}{4} - \frac{2}{3}y^3\right) + \frac{y^2}{2} - (\overline{p})y$$



Cessi's Model

Let p be of the form $p(t) = \overline{p} + p'(t)$ with

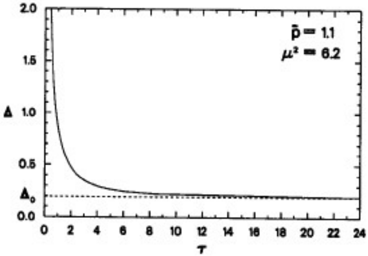
$$p'(t) = \begin{cases} 0 & t \leq 0 \\ \Delta & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$$

Cessi's Model

Once the salinity forcing turns on, y changes according to the integral

$$\int_{y_a}^{y} \frac{d\tilde{y}}{-[1+\mu^2(\tilde{y}-1)^2]\tilde{y}+\overline{p}+\Delta} = \int_0^\tau dt$$

A transition between the two stable states depends on the time over which the forcing is applied.



Cessi's Model

This also shows that a critical forcing amplitude is necessary for the transition, i.e. total volume of fresh water is not the determining factor!

$$\overline{p} + \Delta_0 = \frac{2}{3} + \frac{2}{27}\mu^2(\pm 1 + (1 - 3\mu^{-2})^{3/2})$$

Roberts and Saha

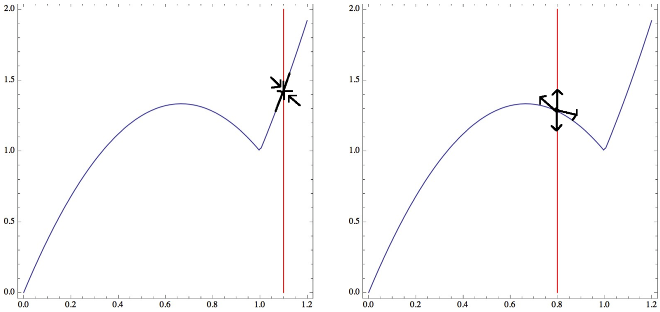
Adding a pulse of fresh water forcing can push Stommel's model into different stable states. What if the salinity forcing is more continuous?

$$
\begin{aligned}
x' &= 1 - x - \varepsilon A|x-y|x \\
y' &= \varepsilon(\mu - y - A|x-y|y) \\
\mu' &= \varepsilon\delta(1 + ax - by)
\end{aligned}
$$

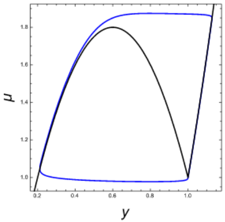
Roberts and Saha

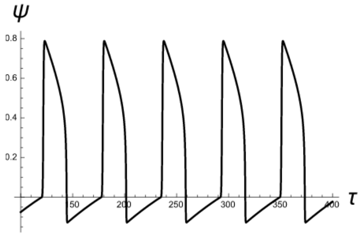
On $x = 1$,

$$\begin{aligned} \dot{y} &= \mu - y - A|1-y|y \\ \dot{\mu} &= \delta_0(\lambda - y) \end{aligned}$$

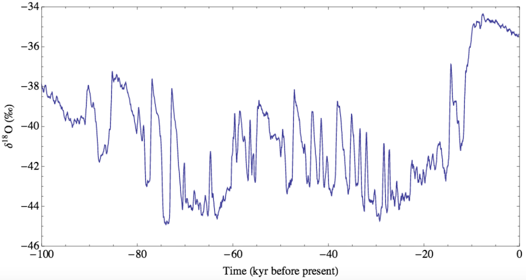


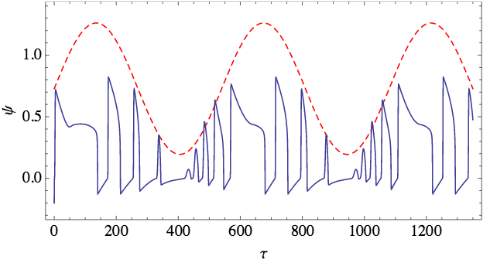
Roberts and Saha





Dansgaard-Oeschger Events





Conclusions

- Ocean box models can have important implications for climate science.
- Ocean box models can have some really cool and complicated math.

References

Stommel, Henry. "Thermohaline convection with two stable regimes of flow." Tellus A 13.2 (2011).

Cessi, Paola. "A simple box model of stochastically forced thermohaline flow." Journal of physical oceanography 24.9 (1994): 1911-1920.

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