Using multiple time-scales to understand Dansgaard-Oeschger events

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Outline

- Introduction
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- Extending the Model
- Forcing and Amplitude Modulation (and Canards!)
Collaborators and Acknowledgments

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• Dissertation advisor: Chris Jones
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Dansgaard-Oeschger Events

- Pulses of abrupt warming over last 100 kyr
- Up to 10 °C warming over a few decades
- Slower “cooling phase”
- Average period 1.5 kyr → internal climate mechanism
- Most intense in North Atlantic, but effects felt globally (at least as far as China)
- Hypothesized mechanism: reversal of AMOC
The Data

Figure: Oxygen isotope data from Greenland (NGRIP).
Bistability of the MOC

Figure: North Atlantic meridional overturning circulation (MOC).
Figure: Schematic of Stommel’s model.
Stommel

The model is:

\[
\frac{dT_e}{dt} = R_T (T_{ea} - T_e) + |\psi|(T_p - T_e) \\
\frac{dT_p}{dt} = R_T (T_{pa} - T_p) + |\psi|(T_e - T_p) \\
\frac{dS_e}{dt} = R_S (S_{ea} - S_e) + |\psi|(S_p - S_e) \\
\frac{dS_p}{dt} = R_S (S_{pa} - S_p) + |\psi|(S_e - S_p) \\
\psi = \psi_0 \left( \frac{\rho_p - \rho_e}{\rho_0} \right),
\]

(1)

where:

- \( T \)'s denote temperatures, \( S \)'s denote salinities
- Subscripts: \( a \) - atmosphere, \( e \) - equator, \( p \) - pole
- \( \psi \) - transport (advection, circulation strength)
- Density \( \rho_i = \rho_0[1 - \alpha(T_i - T_0) + \beta(S_i - S_0)] \).
The change of variables:

\[
T = T_e - T_p, \quad S = S_e - S_p,
\]

\[
T^a = T_e^a - T_p^a, \quad S^a = S_e^a - S_p^a,
\]

\[
X = T_e + T_p \quad Y = S_e + S_p,
\]

turns the model into

\[
\frac{d}{dt} T = R_T (T^a - T) - 2|\psi| T
\]

\[
\frac{d}{dt} S = R_S (S^a - S) - 2|\psi| S
\]

\[
\psi = \psi_0 (\alpha T - \beta S),
\]

where \(X, Y\) decouple.
To non-dimensionalize the system, set

\[ x = \frac{T}{T^a}, \quad y = \frac{\beta S}{\alpha T^a}, \quad \tau = R_S t, \quad \mu = \frac{\beta S^a}{\alpha T^a}, \quad A = \frac{2\psi_0 \alpha T^a}{R_S}. \]

Then the model becomes:

\[
\begin{align*}
\varepsilon \dot{x} &= 1 - x - \varepsilon A|x - y|x \\
\dot{y} &= \mu - y - A|x - y|y, 
\end{align*}
\]

where

\[ \varepsilon = \frac{R_S}{R_T} \ll 1. \]

**GSPT:** The set \( \{x = 1\} \) is attracting and normally hyperbolic. For \( \varepsilon \ll 1 \), solutions will end up within \( \mathcal{O}(\varepsilon) \) of \( x = 1 \).
Reduced Flow

Dynamics of the full system can be understood by the 1D system:

$$\dot{y} = \mu - y - A|1 - y|y.$$  (4)

Equilibria occur at

$$\mu = \begin{cases} 
(1 + A)y - Ay^2 & \text{for } y < 1 \\
(1 - A)y + Ay^2 & \text{for } y > 1 
\end{cases}$$ (5)

Figure: Graphs of equilibria for (a) $A < 1$ and (b) $A > 1$. 

**Figure**: Graphs of equilibria for (a) $A < 1$ and (b) $A > 1$. 
Hysteresis

Figure: Bifurcation diagram for reduced equation (dashed), with a hysteresis loop (solid black) overlay. $\psi = \alpha \psi_0 (1 - y)$.

Problem: what causes $\mu$ to vary?
Mathematics? Climate?
Hysteresis vs. Relaxation Oscillations

**Hysteresis:**

\[ \dot{y} = \mu(t) - y - A|1 - y|y \]

Oscillations result from slowly varying parameter \( \mu \). Period and amplitude determined by external forces.

**Relaxation oscillation:**

\[ \dot{y} = \mu - y - A|1 - y|y \]

\[ \dot{\mu} = \delta g(x, y, \mu; \lambda) \]

Oscillations result from periodic orbit, \( \mu \) changes based on state variables. Period and amplitude sensitive to parameters of the system.
Figure: Critical manifold $\mu = A|1 - y|y$ when $A > 1$. Globally attracting equilibrium when $\lambda = 1.05$. 
Figure: Critical manifold $\mu = A|1 - y|y$ when $A > 1$. Continuum of homoclinic orbits when $\lambda = 1$. 

Hopf Bifurcation
Hopf Bifurcation

Figure: Critical manifold $\mu = A|1 - y|y$ when $A > 1$. Unique periodic orbit when $\lambda = 0.95$. 

Canard cycles

(a) Singular canard cycle.

(b) Singular maximal canard.

(c) Singular canard with head.

(d) A duck!

Figure: Singular Canards
Canard Explosion in Smooth Systems

**Figure:** Canard explosion for fixed $\varepsilon > 0$. Figure from Krupa and Szmolyan (2001)
Canards in PWSC Systems?

(a) Canards when Hopf bifurcation for $\lambda < 0$.

(b) Canards when Hopf bifurcation occurs for $\lambda > 0$.

(c) Canards when Hopf bifurcation occurs at $\lambda = 0$.

(d) Super-explosion.
Extending Stommel’s Model

Including $\mu$ as slow state variable:

\[
\begin{align*}
    x' &= 1 - x - \varepsilon A|x - y|x \\
    y' &= \varepsilon(\mu - y - A|x - y|y) \\
    \mu' &= \varepsilon \delta(1 + ax - by).
\end{align*}
\]  

(6)

Question: what climate component is modeled by $\mu$?
GSPT and GSPT

Critical manifold \( \{x = 1\} \) is still attracting. However, the reduced problem,

\[
\begin{align*}
\dot{y} &= \mu - y - A|1 - y|y \\
\dot{\mu} &= \delta(1 + a - by),
\end{align*}
\]

(7)

is now itself a fast/slow system which is analyzed using GSPT. The critical manifold of (7) is given by

\[
M_0 = \{\mu = y + A|1 - y|y\}.
\]

(8)

Note that this is precisely the curve of equilibria from (3) the dimensionless Stommel model earlier.
To simplify the analysis, rewrite (7) as

\[
\begin{align*}
\dot{y} &= \mu - y - A|1 - y|y \\
\dot{\mu} &= \delta_0(\lambda - y),
\end{align*}
\]

where \(\delta_0 = \delta b\) and \(\lambda = (1 + a)/b\).
Oscillations in the Extended Stommel Model

(a) Stable periodic orbit when $A = 5$, $\lambda = 0.8$, and $\delta = 0.1$

(b) Time series for $\psi$ for the trajectory in (a)

(c) Canard trajectory when $A = 1.1$, $\lambda = 0.995$, and $\delta_0 = 0.01$.

(d) Super-explosion when $A = 1.5$, $\lambda = 0.995$, and $\delta_0 = 0.01$.

Figure: Oscillatory behavior in (9).
Astronomical Forcing

A variation in zonal insolation gradients will naturally affect the atmospheric temperature ($T^a$) and salinity ($S^a$) gradients. In system (9), the inclusion of orbital forcing implies that parameters $A$ and $\lambda$ become time-dependent. The new system becomes

$$
\dot{y} = \mu - y - A(\tau)|1 - y|y
$$

$$
\dot{\mu} = \delta_0(\lambda(\tau) - y),
$$

where

$$
A(\tau) = \bar{A} + p \sin \omega \tau
$$

$$
\lambda(\tau) = \bar{\lambda} + q \sin \omega (\tau - \theta).
$$
(a) Trajectory of the forced system in phase space.

(b) Time series for $\psi$ (solid) with the scaled obliquity (dashed) variations from the last 100 kyr. The units of time are arbitrary.

Figure: Oscillations in the forced system (10) when $\delta_0 = 0.07$, $\bar{A} = 3.5$, $p = 2.4$, $\bar{\lambda} = 0.8$, $q = 1.99$, $\omega = \pi/270$, and $\theta = 250$. 
Comparison with data

(a) Oxygen isotope data from Greenland (NGRIP).

(b) Time series for $\psi$ (solid) in (7).

**Figure:** Dansgaard-Oeschger cycles in (a) data and (b) a conceptual model (7). The parameters used to generate the times series in (b) are $\delta_0 = 0.1$, $\bar{A} = 3.5$, $p = 2.4$, $\bar{\lambda} = 0.8$, $q = 1.99$, $\omega = \pi/270$, and $\theta = 250$. 