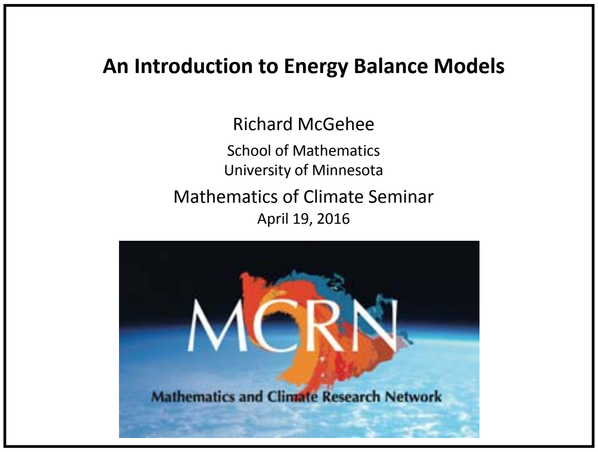

An Introduction to Energy Balance Models

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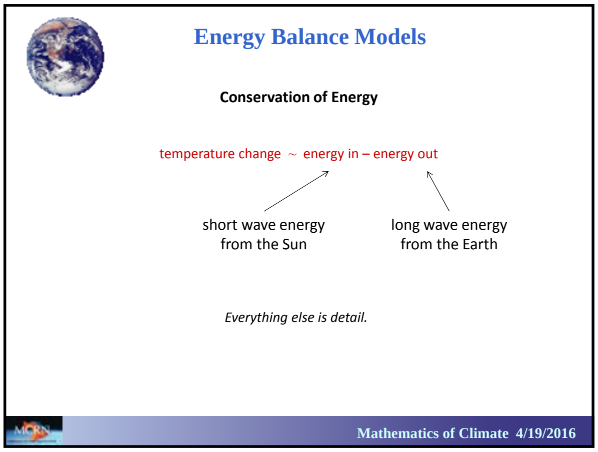
School of Mathematics
University of Minnesota

Mathematics of Climate Seminar
April 19, 2016



Energy Balance Models

Conservation of Energy

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

Everything else is detail.

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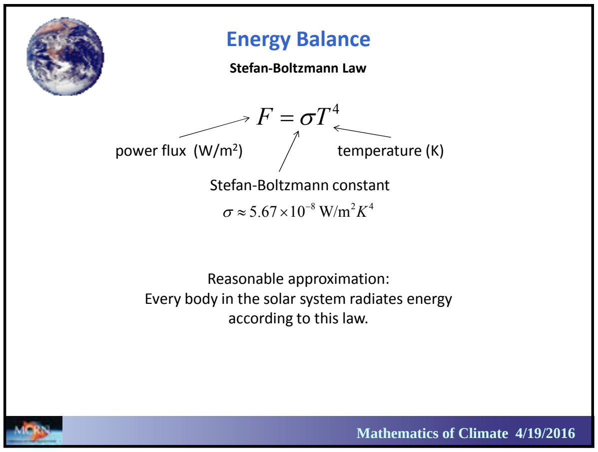

Energy Balance

Stefan-Boltzmann Law

$$F = \sigma T^4$$

power flux (W/m²)

temperature (K)

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2 K^4$$

Reasonable approximation:
Every body in the solar system radiates energy according to this law.

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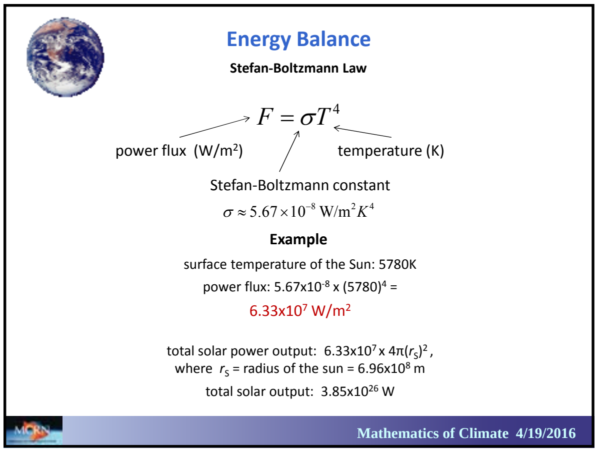

Energy Balance

Stefan-Boltzmann Law

$$F = \sigma T^4$$

power flux (W/m²)

temperature (K)

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8} \text{ W/m}^2 K^4$$

Example

surface temperature of the Sun: 5780K

power flux: $5.67 \text{x} 10^{-8} \text{ x } (5780)^4$ =

$6.33 \text{x} 10^7$ W/m²

total solar power output: $6.33 \text{x} 10^7 \text{ x } 4\pi(r_S)^2$,
where r_S = radius of the sun = $6.96 \text{x} 10^8$ m

total solar output: $3.85 \text{x} 10^{26}$ W

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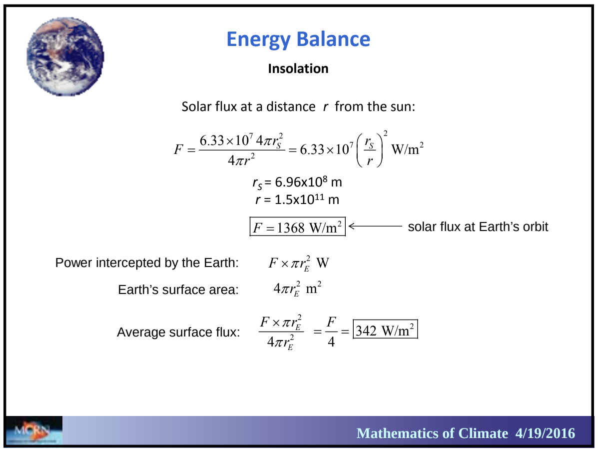

Energy Balance

Insolation

Solar flux at a distance *r* from the sun:

$$F = \frac{6.33 \times 10^7 \, 4\pi r_S^2}{4\pi r^2} = 6.33 \times 10^7 \left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

$r_S = 6.96 \text{x} 10^8$ m
$r = 1.5 \text{x} 10^{11}$ m

$$\boxed{F = 1368 \text{ W/m}^2} \longleftarrow \text{solar flux at Earth's orbit}$$

Power intercepted by the Earth: $F \times \pi r_E^2$ W

Earth's surface area: $4\pi r_E^2$ m²

Average surface flux: $\dfrac{F \times \pi r_E^2}{4\pi r_E^2} = \dfrac{F}{4} = \boxed{342 \text{ W/m}^2}$

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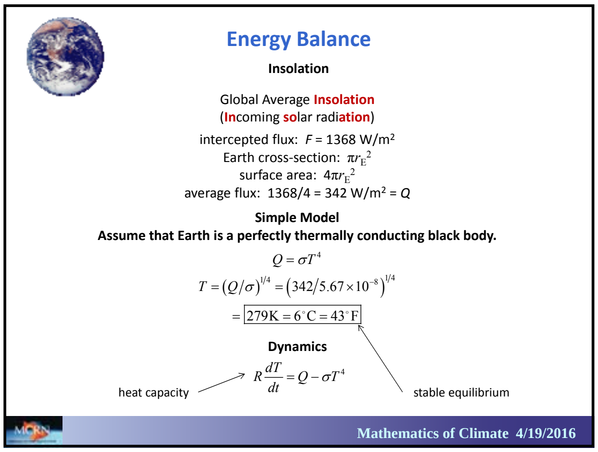

Energy Balance

Insolation

Global Average **Insolation**
(**In**coming **so**lar radi**ation**)

intercepted flux: $F = 1368$ W/m²
Earth cross-section: πr_E^2
surface area: $4\pi r_E^2$
average flux: 1368/4 = 342 W/m² = Q

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = \left(342/5.67 \times 10^{-8}\right)^{1/4}$$

$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity

stable equilibrium

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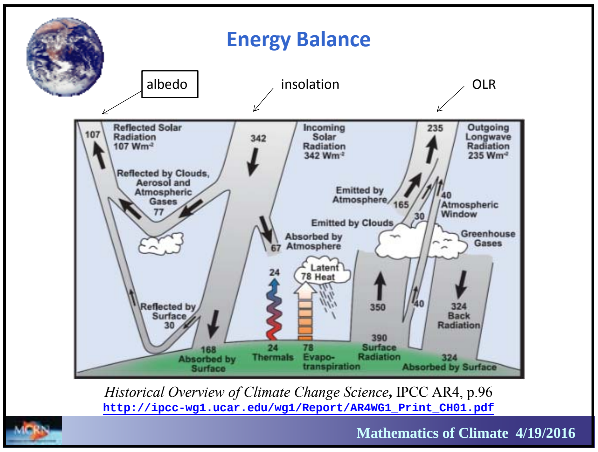
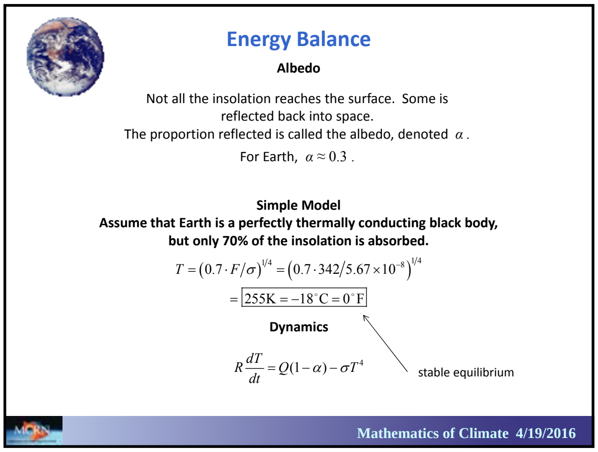


Energy Balance

Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf



Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.

For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

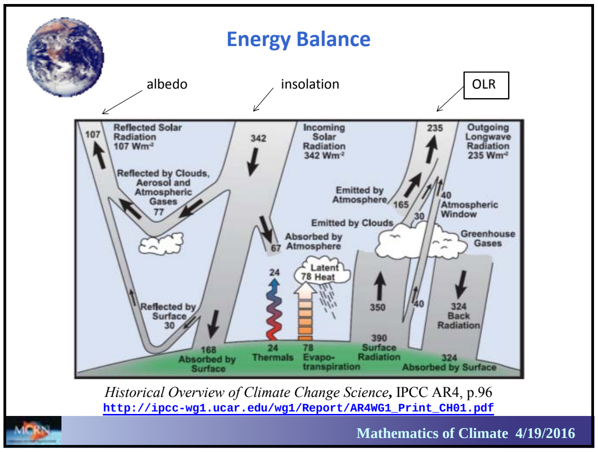
$$T = (0.7 \cdot F/\sigma)^{1/4} = (0.7 \cdot 342/5.67 \times 10^{-8})^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium



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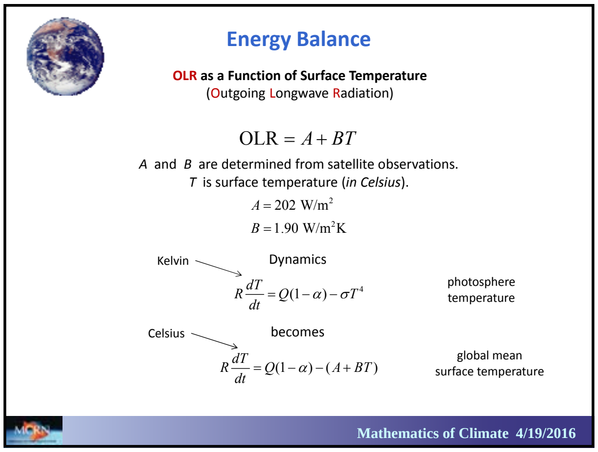


Energy Balance

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} = A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$A = 202 \text{ W/m}^2$

$B = 1.90 \text{ W/m}^2\text{K}$

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

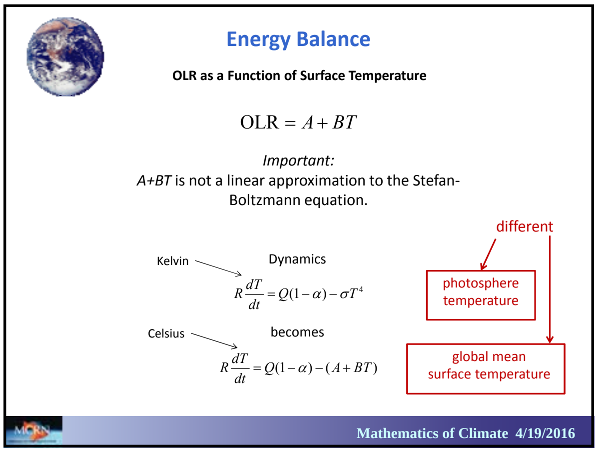
photosphere temperature

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature



Energy Balance

OLR as a Function of Surface Temperature

$$\text{OLR} = A + BT$$

Important:
A+*BT* is not a linear approximation to the Stefan-Boltzmann equation.

different

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

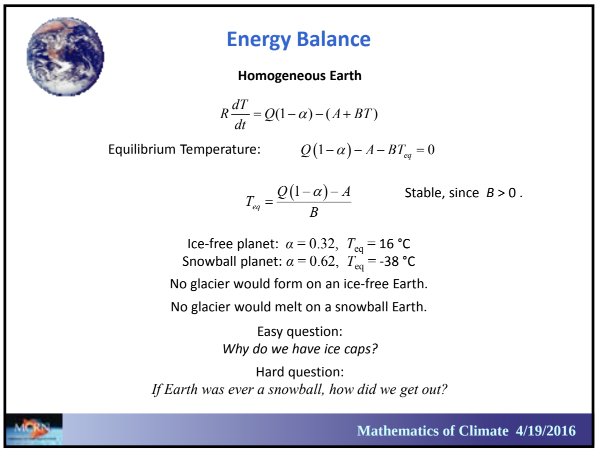
photosphere temperature

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature



Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT_{eq} = 0$

$$T_{eq} = \frac{Q(1-\alpha) - A}{B}$$

Stable, since $B > 0$.

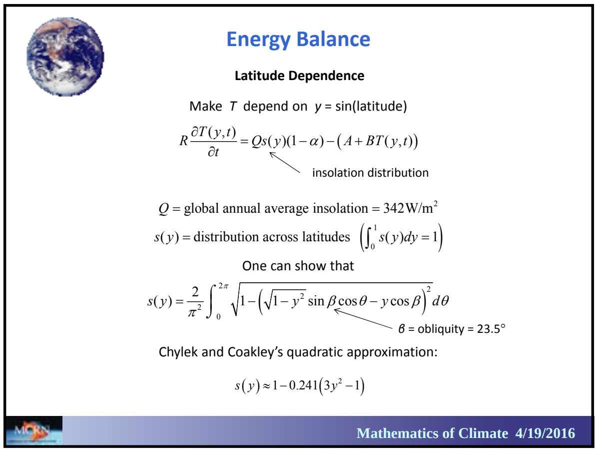
Ice-free planet: $\alpha = 0.32$, T_{eq} = 16 °C
Snowball planet: $\alpha = 0.62$, T_{eq} = -38 °C

No glacier would form on an ice-free Earth.

No glacier would melt on a snowball Earth.

Easy question:
Why do we have ice caps?

Hard question:
If Earth was ever a snowball, how did we get out?



Energy Balance

Latitude Dependence

Make T depend on y = sin(latitude)

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

Q = global annual average insolation = $342\,\mathrm{W/m^2}$

$s(y)$ = distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

One can show that

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

β = obliquity = 23.5°

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2-1\right)$$

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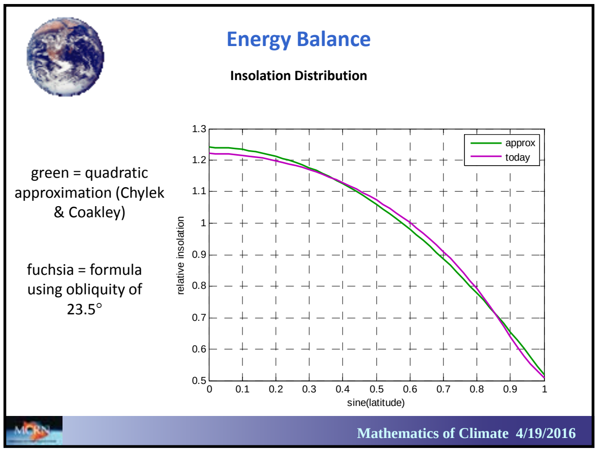

Energy Balance

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°

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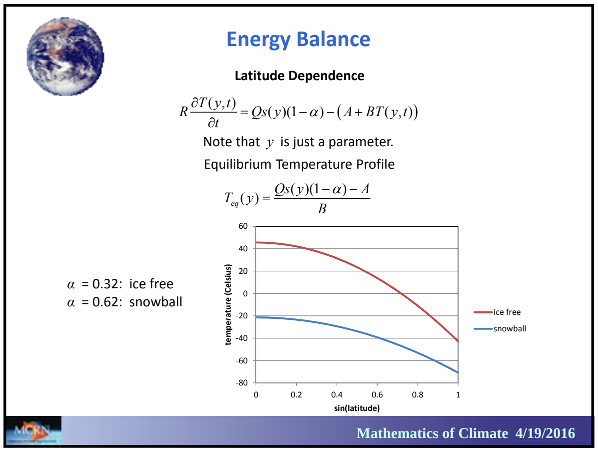

Energy Balance

Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

Note that y is just a parameter.

Equilibrium Temperature Profile

$$T_{eq}(y) = \frac{Qs(y)(1-\alpha) - A}{B}$$

α = 0.32: ice free

α = 0.62: snowball

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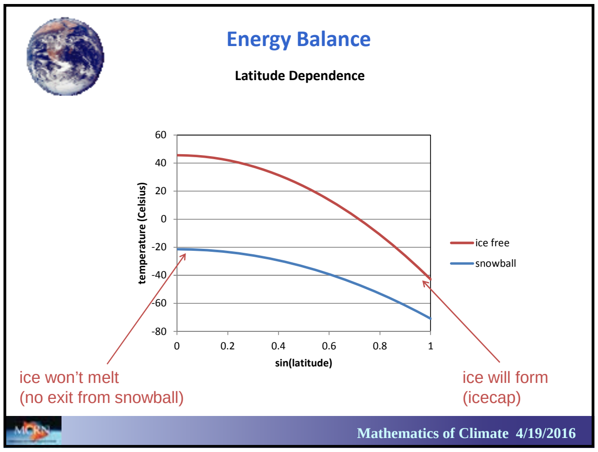

Energy Balance

Latitude Dependence

ice won't melt (no exit from snowball)

ice will form (icecap)

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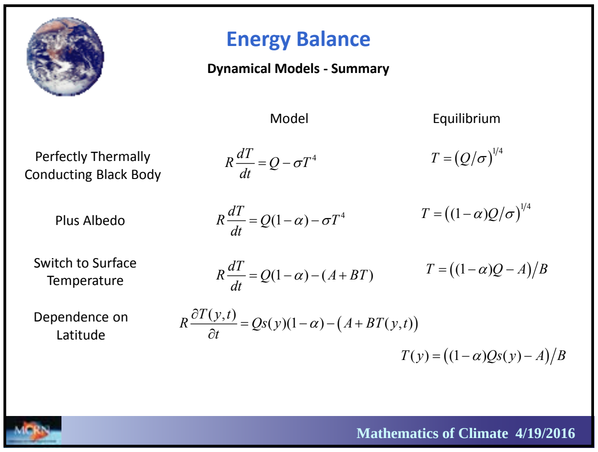

Energy Balance

Dynamical Models - Summary

	Model	Equilibrium
Perfectly Thermally Conducting Black Body	$R\frac{dT}{dt} = Q - \sigma T^4$	$T = \left(Q/\sigma\right)^{1/4}$
Plus Albedo	$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$	$T = \left((1-\alpha)Q/\sigma\right)^{1/4}$
Switch to Surface Temperature	$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$	$T = \left((1-\alpha)Q - A\right)/B$
Dependence on Latitude	$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$	$T(y) = \left((1-\alpha)Qs(y) - A\right)/B$

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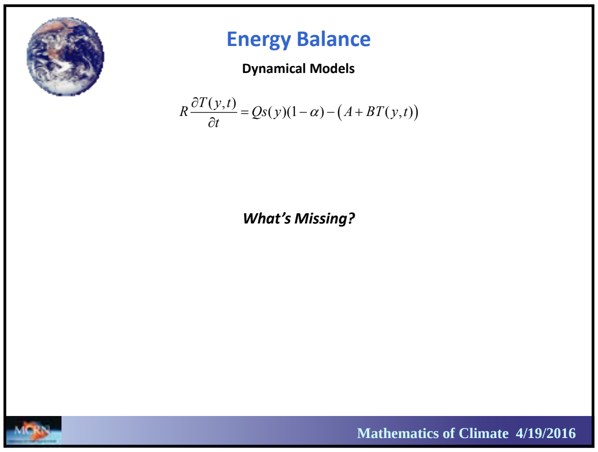

Energy Balance

Dynamical Models

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

What's Missing?

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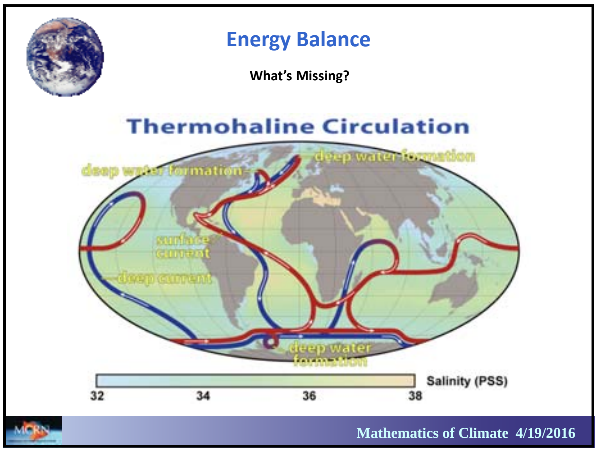

Energy Balance

What's Missing?

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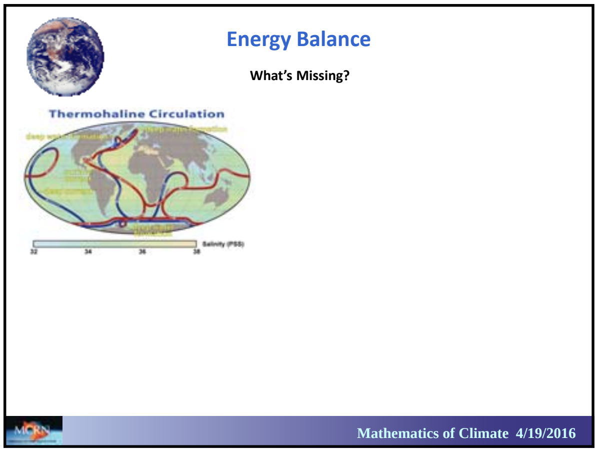

Energy Balance

What's Missing?

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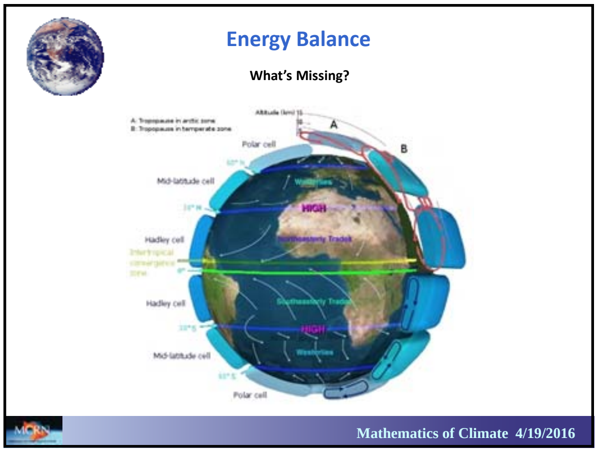

Energy Balance

What's Missing?

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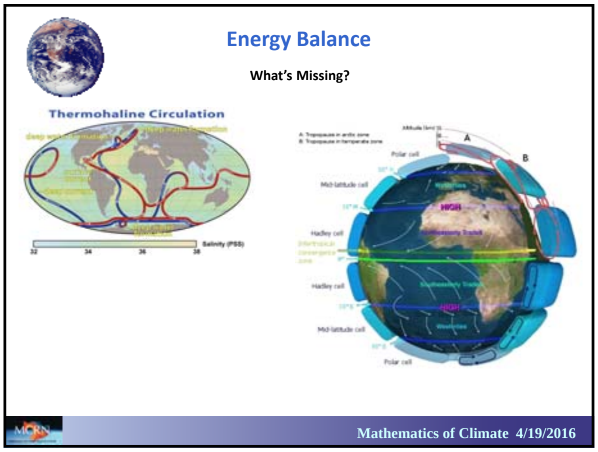

Energy Balance

What's Missing?

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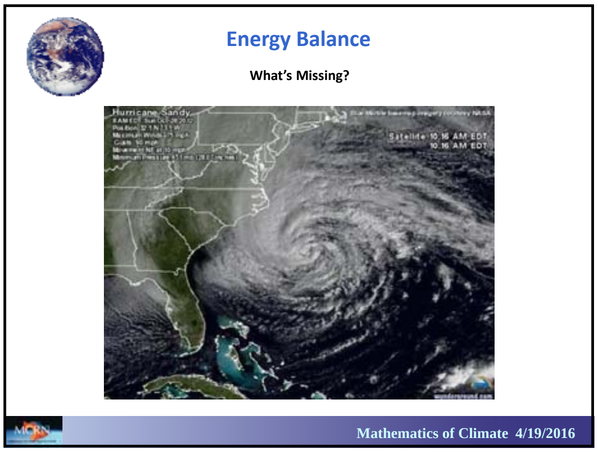

Energy Balance

What's Missing?

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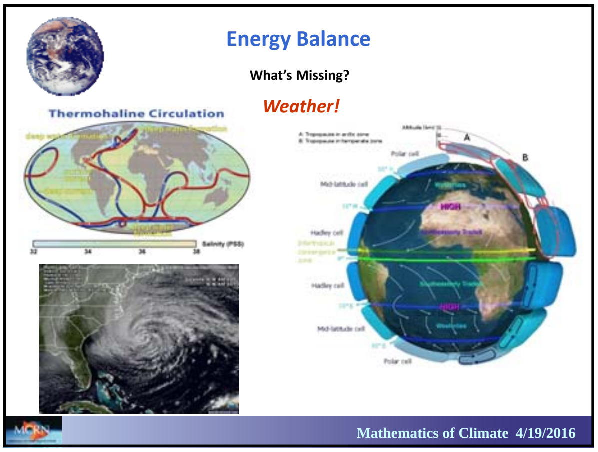

Energy Balance

What's Missing?

Weather!

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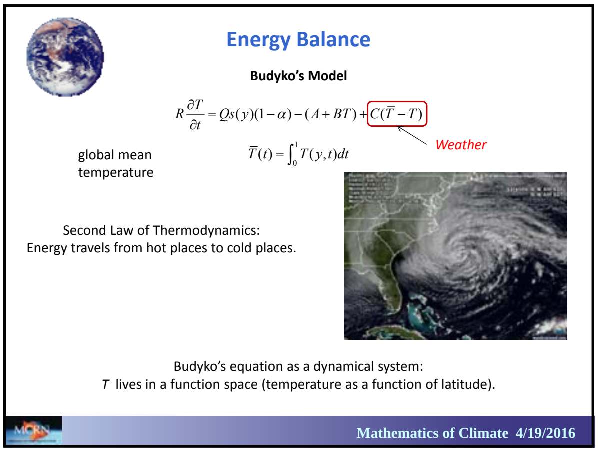

Energy Balance

Budyko's Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + \boxed{C(\overline{T}-T)}$$

Weather

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dt$

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Budyko's equation as a dynamical system:
T lives in a function space (temperature as a function of latitude).

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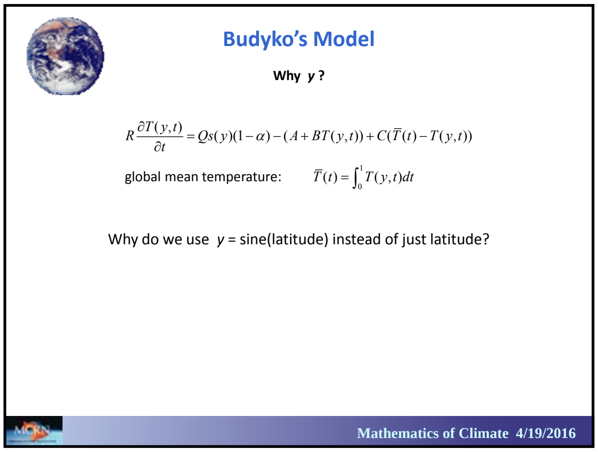

Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A+BT(y,t)) + C(\overline{T}(t) - T(y,t))$$

global mean temperature: $\overline{T}(t) = \int_0^1 T(y,t)dt$

Why do we use y = sine(latitude) instead of just latitude?

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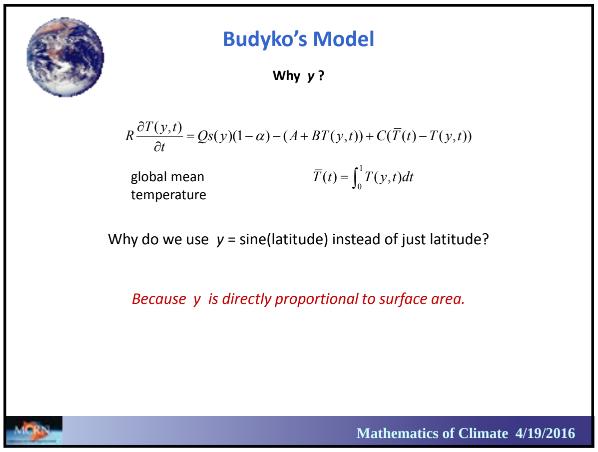

Budyko's Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A+BT(y,t)) + C(\overline{T}(t) - T(y,t))$$

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dt$

Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.

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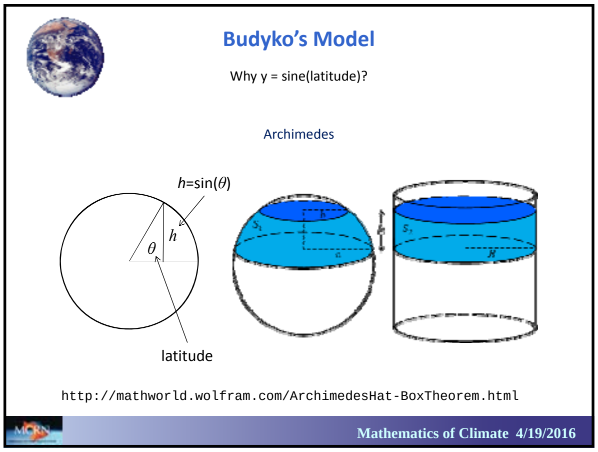

Budyko's Model

Why y = sine(latitude)?

Archimedes

`http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html`

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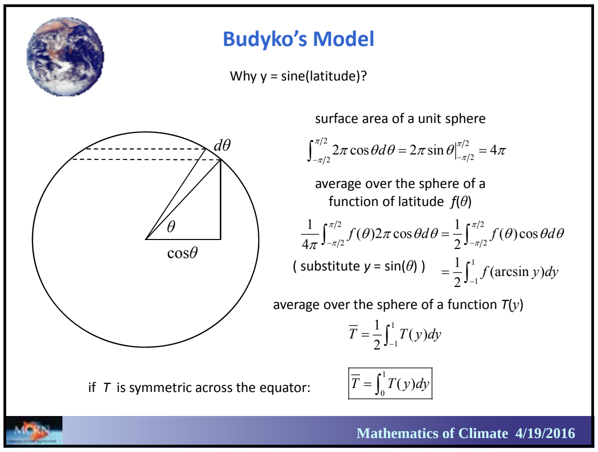

Budyko's Model

Why y = sine(latitude)?

surface area of a unit sphere

$$\int_{-\pi/2}^{\pi/2} 2\pi\cos\theta d\theta = 2\pi\sin\theta\Big|_{-\pi/2}^{\pi/2} = 4\pi$$

average over the sphere of a function of latitude $f(\theta)$

$$\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} f(\theta)2\pi\cos\theta d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} f(\theta)\cos\theta d\theta$$

(substitute $y = \sin(\theta)$) $= \frac{1}{2}\int_{-1}^{1} f(\arcsin y)dy$

average over the sphere of a function $T(y)$

$$\overline{T} = \frac{1}{2}\int_{-1}^{1} T(y)dy$$

if T is symmetric across the equator: $\boxed{\overline{T} = \int_0^1 T(y)dy}$

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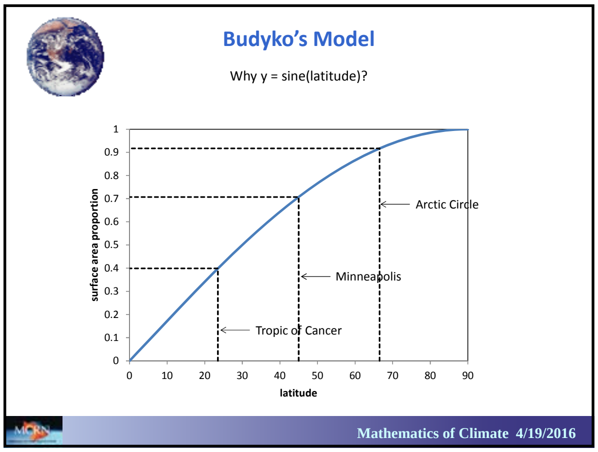

Budyko's Model

Why y = sine(latitude)?

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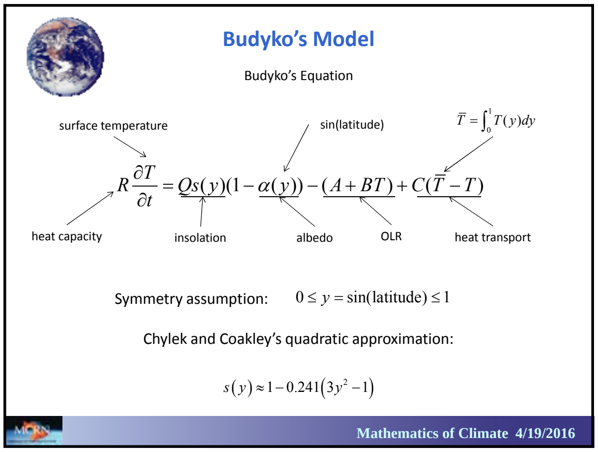


Budyko's Model

Budyko's Equation

$$\underbrace{R}_{\text{heat capacity}}\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\bar{T}-T)}$$

surface temperature

sin(latitude)

$\bar{T} = \int_0^1 T(y)\,dy$

heat capacity

insolation

albedo

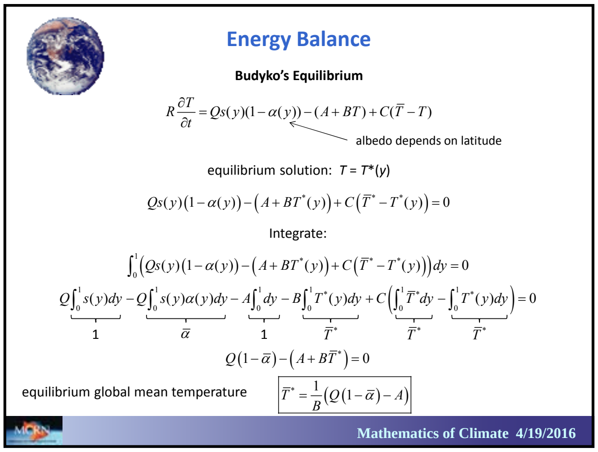
OLR

heat transport

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$



Energy Balance

Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T}-T)$$

albedo depends on latitude

equilibrium solution: $T = T^*(y)$

$$Qs(y)\left(1-\alpha(y)\right) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0$$

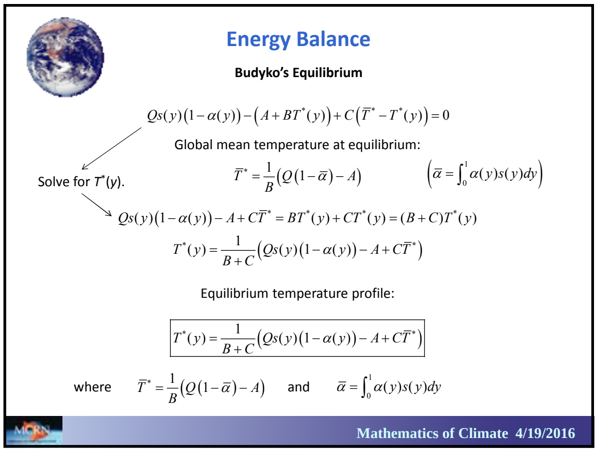
Integrate:

$$\int_0^1 \left(Qs(y)\left(1-\alpha(y)\right) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right)\right)dy = 0$$

$$Q\underbrace{\int_0^1 s(y)dy}_{1} - Q\underbrace{\int_0^1 s(y)\alpha(y)dy}_{\bar{\alpha}} - A\underbrace{\int_0^1 dy}_{1} - B\underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*} + C\left(\underbrace{\int_0^1 \bar{T}^* dy}_{\bar{T}^*} - \underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*}\right) = 0$$

$$Q\left(1-\bar{\alpha}\right) - \left(A + B\bar{T}^*\right) = 0$$

equilibrium global mean temperature $\boxed{\bar{T}^* = \frac{1}{B}\left(Q\left(1-\bar{\alpha}\right) - A\right)}$



Energy Balance

Budyko's Equilibrium

$$Qs(y)\left(1-\alpha(y)\right) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0$$

Global mean temperature at equilibrium:

$$\bar{T}^* = \frac{1}{B}\left(Q\left(1-\bar{\alpha}\right) - A\right) \qquad \left(\bar{\alpha} = \int_0^1 \alpha(y)s(y)dy\right)$$

Solve for $T^*(y)$.

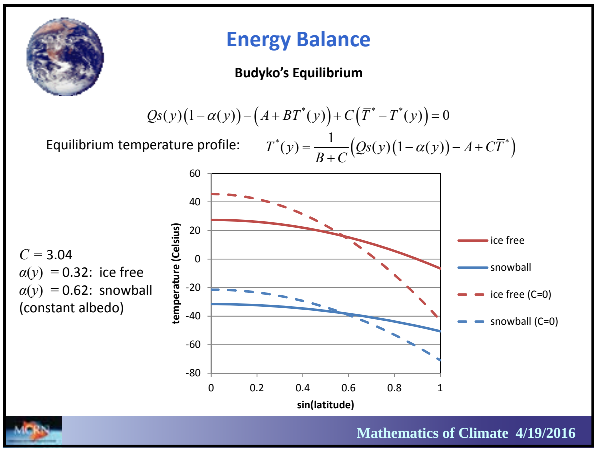
$$Qs(y)\left(1-\alpha(y)\right) - A + C\bar{T}^* = BT^*(y) + CT^*(y) = (B+C)T^*(y)$$

$$T^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y)\right) - A + C\bar{T}^*\right)$$

Equilibrium temperature profile:

$$\boxed{T^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y)\right) - A + C\bar{T}^*\right)}$$

where $\bar{T}^* = \frac{1}{B}\left(Q\left(1-\bar{\alpha}\right) - A\right)$ and $\bar{\alpha} = \int_0^1 \alpha(y)s(y)dy$



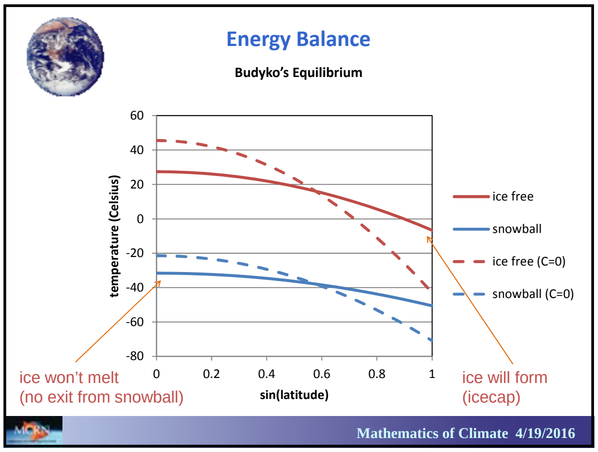
Energy Balance

Budyko's Equilibrium

$$Qs(y)\left(1-\alpha(y)\right) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0$$

Equilibrium temperature profile: $T^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y)\right) - A + C\bar{T}^*\right)$

$C = 3.04$
$\alpha(y) = 0.32$: ice free
$\alpha(y) = 0.62$: snowball
(constant albedo)

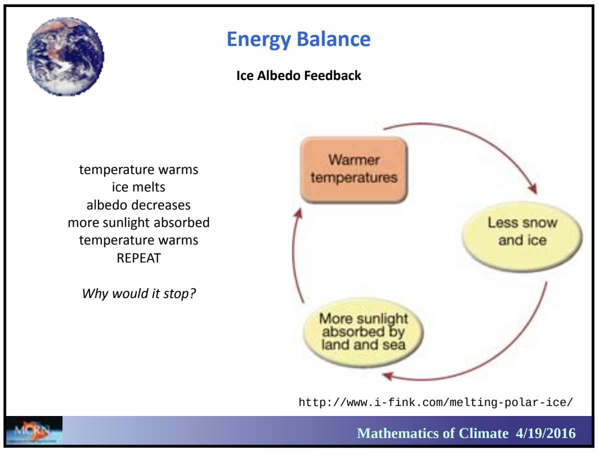


Energy Balance

Budyko's Equilibrium

ice won't melt
(no exit from snowball)

ice will form
(icecap)



Energy Balance

Ice Albedo Feedback

temperature warms
ice melts
albedo decreases
more sunlight absorbed
temperature warms
REPEAT

Why would it stop?

http://www.i-fink.com/melting-polar-ice/

Energy Balance

Ice Albedo Feedback

M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," Tellus XXI, 611-619 , 1969.

temperature warms
ice melts
albedo decreases
more sunlight absorbed
temperature warms
REPEAT

Why would it stop?

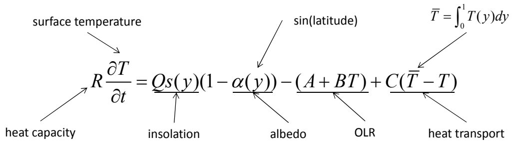
http://www.inenco.org/index_principals.html

Energy Balance

Ice Albedo Feedback

Why would it stop?

Budyko’s Equation



surface temperature

sin(latitude)

$\bar{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = \underline{Qs(y)}(1-\underline{\alpha(y)}) - \underline{(A+BT)} + \underline{C(\bar{T}-T)}$$

heat capacity

insolation

albedo

OLR

heat transport

Energy Balance

Ice Albedo Feedback

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & y < \eta \\ \alpha_2 & y > \eta \end{cases}$$

© AFP/Getty Images

http://galacticconnection.com/wp-content/uploads/2015/04/ice.jpg

Energy Balance

Ice Albedo Feedback

What if the albedo is not constant?

Ice Line Assumption: There is a single ice line at *y=η* between the equator and the pole. The albedo is α_1 below the ice line and α_2 above it.

$$\alpha(y,\eta) = \begin{cases} \alpha_1 & y < \eta \\ \alpha_2 & y > \eta \end{cases} \quad \text{and}$$

Equilibrium condition:

$$Qs(y)\big(1-\alpha(y,\eta)\big) - \big(A + BT_\eta^*(y)\big) + C\big(\bar{T}^* - T_\eta^*(y)\big) = 0$$

Equilibrium solution:

$$T_\eta^*(y) = \frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}_\eta^*\big)$$

where $\bar{T}_\eta^* = \frac{1}{B}\big(Q\big(1-\bar{\alpha}(\eta)\big) - A\big)$ $\quad \left(\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy\right)$

Energy Balance

Ice Albedo Feedback

equilibrium temperature profile:

$$T_\eta^*(y) = \frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}_\eta^*\big), \text{ where } \bar{T}_\eta^* = \frac{1}{B}\big(Q\big(1-\bar{\alpha}(\eta)\big) - A\big)$$

albedo: $\alpha(y,\eta) = \begin{cases} \alpha_1 = 0.32, & y < \eta \\ \alpha_2 = 0.62, & y > \eta \end{cases}$

global albedo: $\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy = \int_0^\eta \alpha_1 s(y)dy + \int_\eta^1 \alpha_2 s(y)dy$

let: $S(\eta) = \int_0^\eta s(y)dy, \quad 1 - S(\eta) = \int_\eta^1 s(y)dy, \quad \text{since} \quad 1 = \int_0^1 s(y)dy$

then: $\bar{\alpha}(\eta) = \alpha_1 S(\eta) + \alpha_2(1 - S(\eta)) = \alpha_2 - (\alpha_2 - \alpha_1)S(\eta) = 0.62 - 0.3S(\eta)$

$$S(\eta) = \int_0^\eta s(y)dy = \int_0^\eta \underbrace{\big(1 - 0.241\big(3y^2 - 1\big)\big)}_{\text{Chylek \& Coakley}} dy = \eta - 0.241\big(\eta^3 - \eta\big)$$

Energy Balance

Ice Albedo Feedback

equilibrium temperature profile:

$$\boxed{T_\eta^*(y) = \frac{1}{B+C}\big(Qs(y)\big(1-\alpha(y,\eta)\big) - A + C\bar{T}_\eta^*\big)}, \text{ where } \boxed{\bar{T}_\eta^* = \frac{1}{B}\big(Q\big(1-\bar{\alpha}(\eta)\big) - A\big)}$$

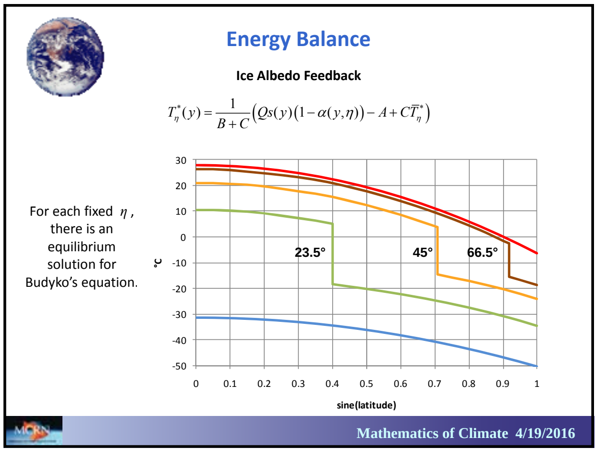
albedo: $\alpha(y,\eta) = \begin{cases} \alpha_1 = 0.32, & y < \eta \\ \alpha_2 = 0.62, & y > \eta \end{cases}$

global albedo: $\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy = \int_0^\eta \alpha_1 s(y)dy + \int_\eta^1 \alpha_2 s(y)dy$

let: $S(\eta) = \int_0^\eta s(y)dy, \quad 1 - S(\eta) = \int_\eta^1 s(y)dy, \quad \text{since} \quad 1 = \int_0^1 s(y)dy$

then: $\boxed{\bar{\alpha}(\eta)} = \alpha_1 S(\eta) + \alpha_2(1 - S(\eta)) = \alpha_2 - (\alpha_2 - \alpha_1)S(\eta) = \boxed{0.62 - 0.3S(\eta)}$

$$\boxed{S(\eta)} = \int_0^\eta s(y)dy = \int_0^\eta \underbrace{\big(1 - 0.241\big(3y^2 - 1\big)\big)}_{\text{Chylek \& Coakley}} dy = \boxed{\eta - 0.241\big(\eta^3 - \eta\big)}$$



Energy Balance

Ice Albedo Feedback

$$T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)$$

For each fixed η, there is an equilibrium solution for Budyko's equation.

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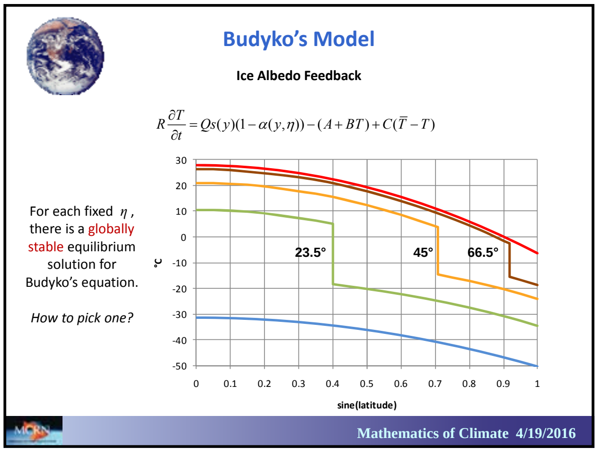

Budyko's Model

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

For each fixed η, there is a globally stable equilibrium solution for Budyko's equation.

How to pick one?

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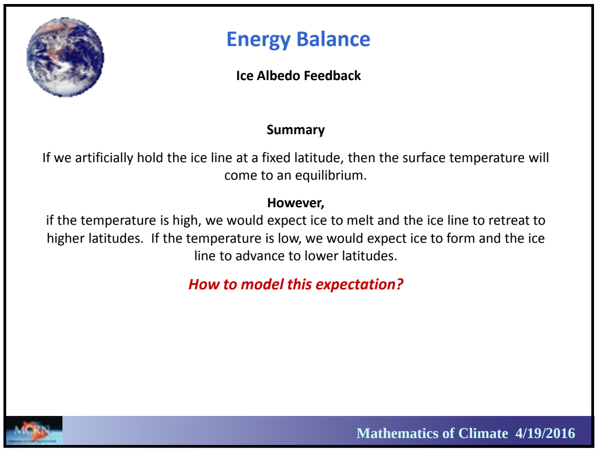

Energy Balance

Ice Albedo Feedback

Summary

If we artificially hold the ice line at a fixed latitude, then the surface temperature will come to an equilibrium.

However,

if the temperature is high, we would expect ice to melt and the ice line to retreat to higher latitudes. If the temperature is low, we would expect ice to form and the ice line to advance to lower latitudes.

How to model this expectation?

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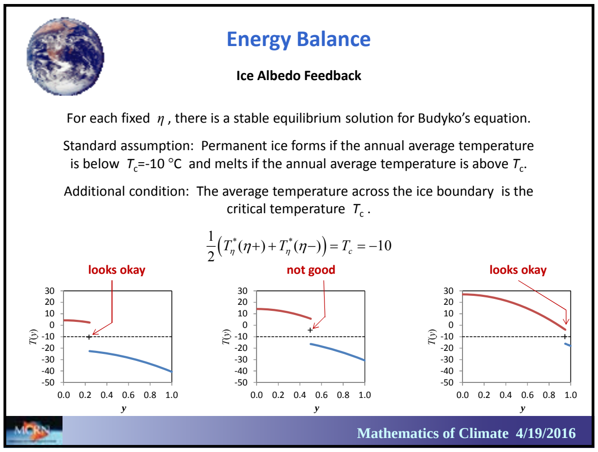

Energy Balance

Ice Albedo Feedback

For each fixed η, there is a stable equilibrium solution for Budyko's equation.

Standard assumption: Permanent ice forms if the annual average temperature is below T_c=-10 °C and melts if the annual average temperature is above T_c.

Additional condition: The average temperature across the ice boundary is the critical temperature T_c.

$$\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$$

looks okay

not good

looks okay

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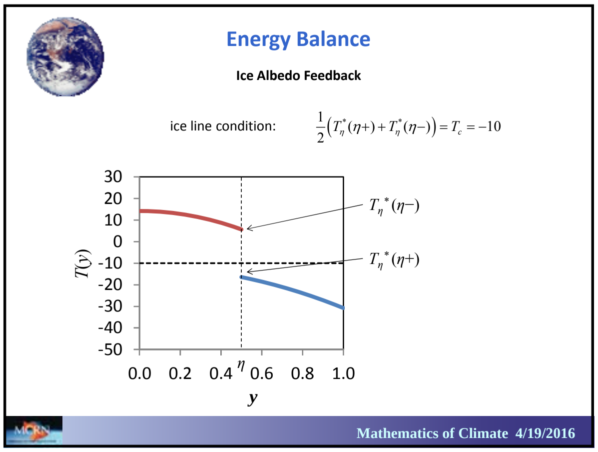

Energy Balance

Ice Albedo Feedback

ice line condition: $\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$

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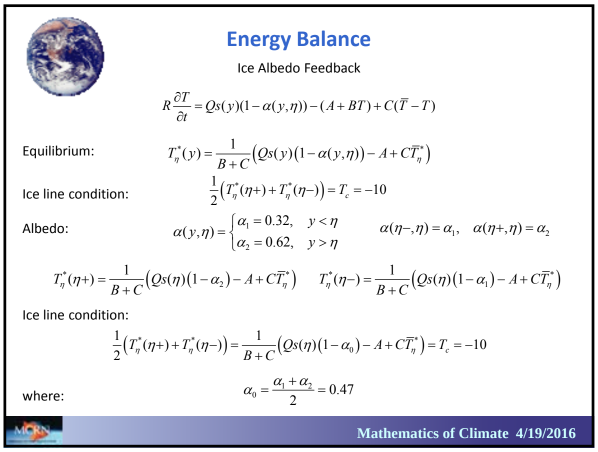

Energy Balance

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\overline{T}-T)$$

Equilibrium: $T_\eta^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\overline{T}_\eta^*\right)$

Ice line condition: $\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = T_c = -10$

Albedo: $\alpha(y,\eta) = \begin{cases} \alpha_1 = 0.32, & y < \eta \\ \alpha_2 = 0.62, & y > \eta \end{cases}$ $\quad \alpha(\eta-,\eta) = \alpha_1, \quad \alpha(\eta+,\eta) = \alpha_2$

$$T_\eta^*(\eta+) = \frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_2\right) - A + C\overline{T}_\eta^*\right) \qquad T_\eta^*(\eta-) = \frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_1\right) - A + C\overline{T}_\eta^*\right)$$

Ice line condition:

$$\frac{1}{2}\left(T_\eta^*(\eta+) + T_\eta^*(\eta-)\right) = \frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right) - A + C\overline{T}_\eta^*\right) = T_c = -10$$

where: $\alpha_0 = \frac{\alpha_1 + \alpha_2}{2} = 0.47$

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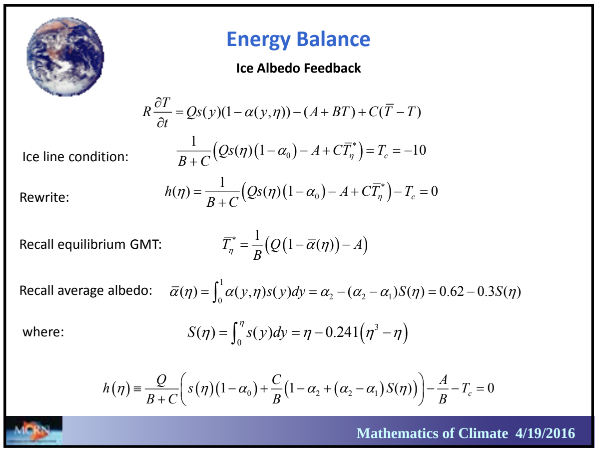


Energy Balance

Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

Ice line condition: $\frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0) - A + C\bar{T}^*_\eta\right) = T_c = -10$

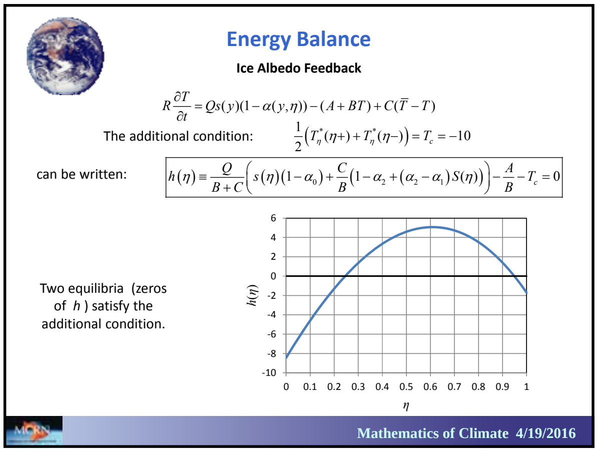
Rewrite: $h(\eta) = \frac{1}{B+C}\left(Qs(\eta)(1-\alpha_0) - A + C\bar{T}^*_\eta\right) - T_c = 0$

Recall equilibrium GMT: $\bar{T}^*_\eta = \frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right) - A\right)$

Recall average albedo: $\bar{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy = \alpha_2 - (\alpha_2-\alpha_1)S(\eta) = 0.62 - 0.3S(\eta)$

where: $S(\eta) = \int_0^\eta s(y)dy = \eta - 0.241\left(\eta^3 - \eta\right)$

$$h(\eta) \equiv \frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)S(\eta)\right)\right) - \frac{A}{B} - T_c = 0$$



Energy Balance

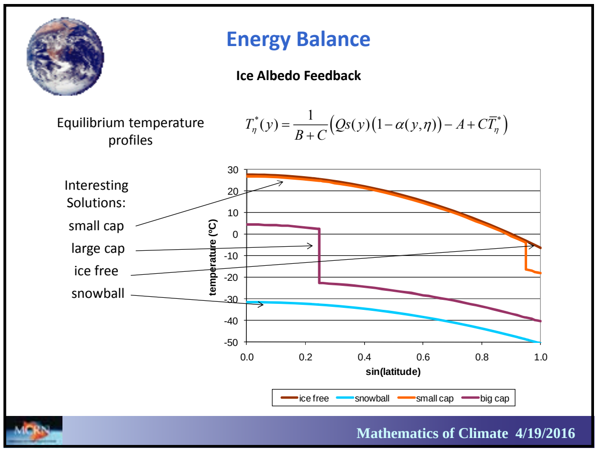
Ice Albedo Feedback

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

The additional condition: $\frac{1}{2}\left(T^*_\eta(\eta+) + T^*_\eta(\eta-)\right) = T_c = -10$

can be written: $\boxed{h(\eta) \equiv \frac{Q}{B+C}\left(s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)S(\eta)\right)\right) - \frac{A}{B} - T_c = 0}$

Two equilibria (zeros of *h*) satisfy the additional condition.

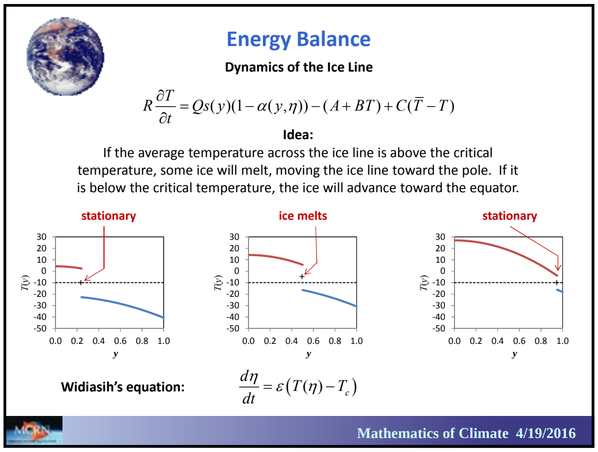


Energy Balance

Ice Albedo Feedback

Equilibrium temperature profiles $T^*_\eta(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y,\eta)\right) - A + C\bar{T}^*_\eta\right)$

Interesting Solutions:



Energy Balance

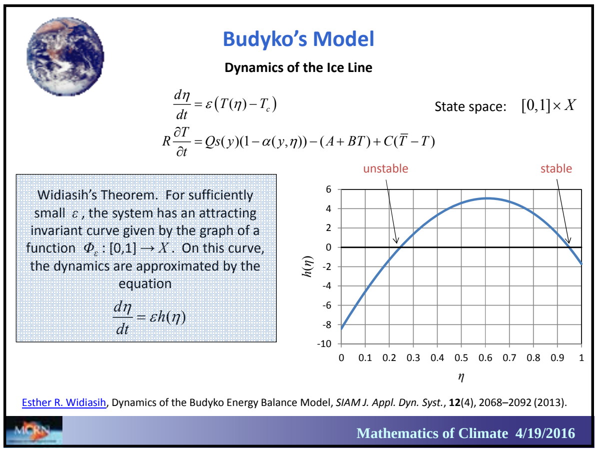
Dynamics of the Ice Line

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

Idea:

If the average temperature across the ice line is above the critical temperature, some ice will melt, moving the ice line toward the pole. If it is below the critical temperature, the ice will advance toward the equator.

Widiasih's equation: $\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$



Budyko's Model

Dynamics of the Ice Line

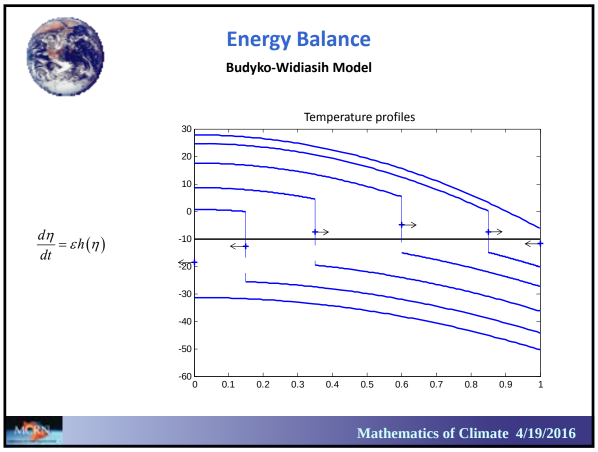
$\frac{d\eta}{dt} = \varepsilon\left(T(\eta) - T_c\right)$ State space: $[0,1]\times X$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) + C(\bar{T}-T)$$

Widiasih's Theorem. For sufficiently small ε, the system has an attracting invariant curve given by the graph of a function $\Phi_\varepsilon : [0,1] \to X$. On this curve, the dynamics are approximated by the equation

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

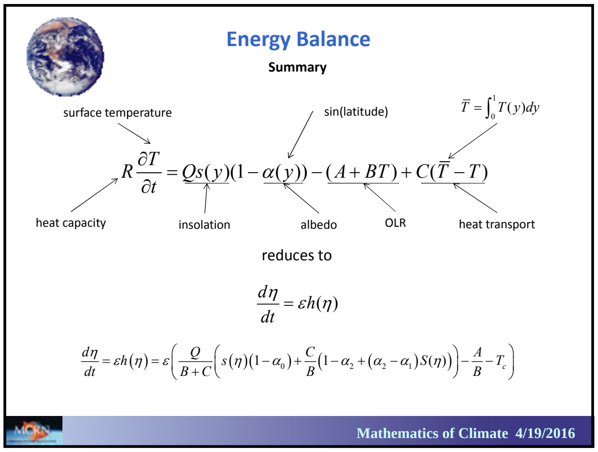
Esther R. Widiasih, Dynamics of the Budyko Energy Balance Model, *SIAM J. Appl. Dyn. Syst.*, **12**(4), 2068–2092 (2013).



Energy Balance

Budyko-Widiasih Model

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$



Energy Balance

Summary

surface temperature

sin(latitude)

$\bar{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T}-T)$$

heat capacity

insolation

albedo

OLR

heat transport

reduces to

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

$$\frac{d\eta}{dt} = \varepsilon h(\eta) = \varepsilon\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha_0)+\frac{C}{B}\left(1-\alpha_2+(\alpha_2-\alpha_1)S(\eta)\right)\right)-\frac{A}{B}-T_c\right)$$

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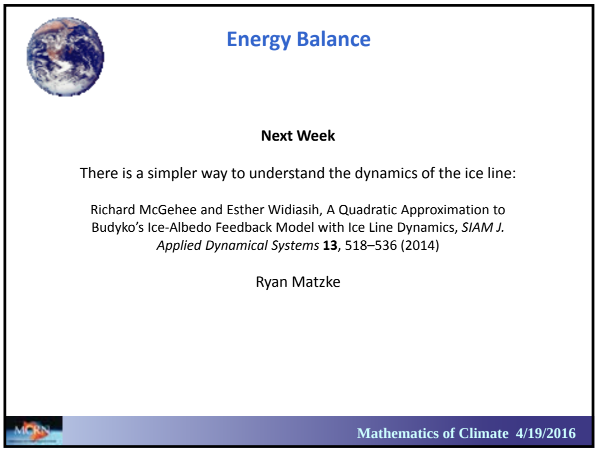

Energy Balance

Next Week

There is a simpler way to understand the dynamics of the ice line:

Richard McGehee and Esther Widiasih, A Quadratic Approximation to Budyko's Ice-Albedo Feedback Model with Ice Line Dynamics, *SIAM J. Applied Dynamical Systems* **13**, 518–536 (2014)

Ryan Matzke

Mathematics of Climate 4/19/2016