Dynamics of Energy Balance Models for Planetary Climate

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Motivation

- Low dimensional climate models are important for understanding the predominant forces affecting the climate of Earth.

- Fly-bys of Pluto and other rocky celestial bodies in our solar system have raised interest in other climates.

Photos from nasa.gov
Energy Balance Models in 1969

- Budyko and Sellers (independently) proposed energy balance models for the Earth (1, 14)
- Wanted to study if another glacial age was possible
- Both models had the same major components: incoming solar radiation, outgoing radiation, and energy transfer:

\[ R \Delta T = Q(y)(1 - \alpha(y)) - (A + BT) + \Gamma(T) \]
Energy Balance Models Today

\[ R \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + BT) + \Gamma(T) \]

where

\[ \Gamma(T(y, \eta)) = -C \left( T(y, \eta) - \int_0^1 T(\gamma, \eta) d\gamma \right). \]

Widiasih introduced an equation for the dynamics of the ice line, \( \eta \), in 2012 (18)

\[ \frac{d}{dt} \eta = \varepsilon(T(y, \eta) - T_c) \]
“Quadratic Approximation”

In (9) McGehee and Widiasih consider the Budyko-type equation for \( y, \eta \in [0, 1] \)

\[
\frac{\partial}{\partial t} T = \frac{1}{R} \left( Q_s(y)(1 - \alpha(\eta, y)) - (A + BT(y, \eta)) - C\left(T(y, \eta) - \bar{T}\right) \right)
\]

with dynamic ice line

\[
\dot{\eta} = \rho(T(y, \eta) - T_c)
\]

and piecewise constant albedo function

\[
\alpha(y, \eta) = \begin{cases} 
\alpha_w & y < \eta \\
\alpha_0 & y = \eta \\
\alpha_i & y > \eta 
\end{cases}
\]

where

\[
\alpha_0 = \frac{\alpha_w + \alpha_i}{2}.
\]
The equilibrium temperature profile can be written

\[
T_\eta^*(y) = \begin{cases} 
\frac{1}{B+C}(Qs(y)(1-\alpha_w) - A + C \overline{T}_\eta^*) & y < \eta \\
\frac{1}{B+C}(Qs(\eta)(1-\alpha_0) - A + C \overline{T}_\eta^*) & y = \eta \\
\frac{1}{B+C}(Qs(y)(1-\alpha_i) - A + C \overline{T}_\eta^*) & y > \eta 
\end{cases}
\]

which motivates the four-dimensional function space \( X \) whose elements are of the form

\[
T(y) = \begin{cases} 
w_0 + \frac{1}{2} z_0 + (w_2 + \frac{1}{2} z_2) p_2(y) & y < \eta \\
w_0 + w_2 p_2(\eta) & y = \eta \\
w_0 - \frac{1}{2} z_0 + (w_2 - \frac{1}{2} z_2) p_2(y) & y > \eta 
\end{cases}
\]
Reformulating the $\partial_t T$ equation in this function space gives

$$R\dot{w}_0 = Q(1 - \alpha_0) - A - Bw_0 + C \left( (\eta - \frac{1}{2})z_0 + z_2 \int_0^\eta p_2(y)dy \right)$$

$$R\dot{z}_0 = Q(\alpha_i - \alpha_w) - (B + C)z_0$$

$$R\dot{w}_2 = Qs_2(1 - \alpha_0) - (B + C)w_2$$

$$R\dot{z}_2 = Qs_2(\alpha_i - \alpha_w) - (B + C)z_2$$

and the ice line equation becomes

$$R\dot{\eta} = \rho \left( w_0 - \frac{Q(1 - \alpha_0)}{B + C} s_2 p_2(\eta) + T_c \right).$$

Write

$$R\dot{w}_0 = -B(w_0 - F(\eta))$$

$$R\dot{\eta} = -\rho(w_0 - G(\eta)).$$
\[(\eta, w_0) \text{ Phase Space}\]

\[
R \dot{w}_0 = -B(w_0 - F(\eta)) \\
R \dot{\eta} = -\rho(w_0 - G(\eta))
\]
The Jormungand Model

Define the function

\[ \delta(\eta) = \begin{cases} 
-\eta + .35, & \eta < .35 \\
0, & \eta \geq .35 
\end{cases} \]

which represents the extent of the bare ice and the Jormungand albedo function

\[ \alpha_J(y, \eta) = \frac{\alpha_s + \alpha_w}{2} + \frac{\alpha_i - \alpha_w}{2} \tanh(M(y - \eta)) \]

\[ + \frac{\alpha_s - \alpha_i}{2} \tanh(M(y - (\eta + \delta(\eta)))) \]
Widiasih’s Theorem still applies with this albedo function.

There is a locally attracting invariant manifold

\[ P_j^* = \{(\Phi_j^*(\eta), \eta) : \eta \in \mathbb{R}\} \]

within \( O(\epsilon) \) of the manifold of fixed points

\[ T_j^* = \{(T_j^*(y, \eta), \eta) : \eta \in \mathbb{R}\} \]
Jormungand Bifurcation in the Greenhouse Gas Parameter
Actual distribution can be found using orbital parameters (as seen in (5, 8, 17)):

\[ s(y, \beta) = \frac{2}{\pi^2} \int_{0}^{2\pi} \sqrt{1 - \left(\sqrt{1 - y^2} \sin \beta \sin \gamma - y \cos \beta\right)^2} \, d\gamma \]

We use Legendre approximations in EBMs because the above integral doesn’t have a closed form expression. Instead use

\[ s(y, \beta) = \sum_{n=0}^{\infty} s_{2n}(\beta)p_{2n}(y) \]
Write the $2n$-th degree Legendre polynomial as

$$p_{2n}(y) = \sum_{k=0}^{n} a_{2k}y^{2k}$$

and

$$s_{2n}(\beta) = P_{2n} \sum_{k=0}^{n} a_{2k} c_{2k}(\beta).$$

where

$$c_{2k}(\beta) = \sum_{j=0}^{k} \binom{2k}{2j} \frac{(\sin \beta)^{2(k-j)}(\cos \beta)^{2j}}{\pi^2} \left(\int_{-\pi/2}^{\pi/2} (\cos \hat{\phi})^{2(k+1-j)}(\sin \hat{\phi})^{2j} d\hat{\phi}\right) \left(\int_{0}^{2\pi} (\cos \hat{\theta})^{2(k-j)} d\hat{\theta}\right)$$
Insolation Distribution on Rapidly Spinning Planets

Earth

Pluto

(a)

(b)

(c)

sine of latitude

-1.0 -0.5 0.0 0.5 1.0

-1.0 -0.5 0.0 0.5 1.0

-1.0 -0.5 0.0 0.5 1.0

-1.0 -0.5 0.0 0.5 1.0

Actual
Approximation

-1.0 -0.5 0.0 0.5 1.0

-1.0 -0.5 0.0 0.5 1.0

-1.0 -0.5 0.0 0.5 1.0

-1.0 -0.5 0.0 0.5 1.0

1.00
1.05
1.10

1.00
1.05
1.10

1.00
1.05
1.10

1.00
1.05
1.10

Actual
Approximation
Small Integer Spin-Orbit Resonances

Mean annual insolation distributions for obliquity $\beta = 60^\circ$ and eccentricity $e = .2$:

A. Dobrovolskis, “Insolation on exoplanets with eccentricity and obliquity.”
Open Questions about Insolation

- Can we quantify when we can use the “rapidly spinning planet” method/formula and still have error less than $\tau$ in our approximation?

- Can we find a closed form expression for insolation on planets with small integer resonances?
Open Questions about Insolation

- For a reasonable range of parameter values, the Budyko map for Pluto doesn’t have any nontrivial stable fixed points. Why?

  - Could Pluto’s “upside down” insolation be playing a factor?

  - Is it because Pluto’s insolation is relatively flat?

![Graphs showing sine of latitude](a) (b) (c)
Open Questions about Ice Lines

- Reformulate model to accommodate ice planets
  - Is more than one ice present and how do we account for different albedos?
  - How are the ices situated on the surface? Are ices mixed? Are they layered?

From nasa.gov
Open Questions about Ice Lines

- Remove symmetry assumptions?

- Investigate four ice lines \((\eta_{SP}, \eta_{SE}, \eta_{NE}, \eta_{NP})\) with the properties

\begin{itemize}
  \item[(i)] \(\eta_{SP}, \eta_{SE}, \eta_{NE}, \eta_{NP} \in [-1, 1]\).
  \item[(ii)] \(-1 \leq \eta_{SP} \leq \eta_{SE} \leq \eta_{NE} \leq \eta_{NP} \leq 1\).
  \item[(iii)] Ice is located between \(\eta_{SP}\) and \(\eta_{SE}\) and between \(\eta_{NE}\) and \(\eta_{NP}\).
\end{itemize}

- Initial investigations into this case show potential oscillations in the ice line dynamics.
Other Open Questions Concerning EBMs

- How do we accurately model a planet whose atmosphere “freezes out” (as Pluto’s might)?

- Is the diffusion model more accurate for a planet with no oceans? For a planet without an atmosphere?
  
  - In which cases do we get the same results with both models?
  
  - In which cases do we get different results?


