An Introduction to Budyko’s Model

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Conservation of Energy

\[
\text{temperature change} \sim \text{energy in} - \text{energy out}
\]

short wave energy from the Sun

long wave energy from the Earth

Everything else is detail.

Budyko’s Model

Dynamical Models

Add Heat Transport

\[
\frac{\partial T}{\partial t} = \left[ b(y)(1-a(y)) - (A + BT) + C(T - T_g) \right]
\]

global mean temperature \( T_g = \int T(y) \, dy \)

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Equilibrium temperature profile?

Budyko’s Equation

Dynamical Models

Perfectly Thermally Conducting Black Body

\[
\frac{\partial T}{\partial t} = Q - \sigma T^4
\]

\( T = (\sigma a)^{\frac{1}{4}} \)

Plus Albedo

\[
\frac{\partial T}{\partial t} = Q(1-a) - \sigma T^4
\]

\( T = ((1-a)\sigma a)^{\frac{1}{4}} \)

Switch to Surface Temperature

\[
\frac{\partial T}{\partial t} = Q(1-a) - (A + BT)
\]

\( T = ((1-a)\sigma a - A)^{\frac{1}{2}} \)

Dependence on Latitude

\[
\frac{\partial T(y)}{\partial t} = Q(y)(1-a(y)) - (A + BT(y))
\]

\( T(y) = ((1-a(y)\sigma a - A)^{\frac{1}{2}} \)

Albedo depends on latitude

\( 0 \leq y = \sin (\text{latitude}) \leq 1 \)

Chylek and Coakley’s quadratic approximation:

\( a(y) = 1 - 0.241(y^2 - 1) \)

Budyko’s Model

Dynamical Models

Budyko’s Equilibrium

\[
\frac{\partial T}{\partial t} = b(y)(1-a(y)) - (A + BT) + C(T - T_g)
\]

Equilibrium solution:

\( T = T_g \)

\( (b(y)(1-a(y)) - (A + BT(y)) + C(T - T_g) = 0 \)

Integrate:

\[
\int_1^0 \left[ b(y)(1-a(y)) - (A + BT(y)) + C(T - T_g) \right] \, dy = 0
\]

\( \int_1^0 Q(y) \, dy - \frac{1}{A} \int_1^0 T(y) \, dy = 0 \)

\( \int_1^0 \left[ Q(y)(1-a(y)) - (A + BT(y)) + C(T - T_g) \right] \, dy = 0 \)

Equilibrium global mean temperature

\( T_g = \frac{1}{1-a} \int_1^0 T(y) \, dy \)
Budyko's Equilibrium

Global mean temperature at equilibrium:
\[ T^{*} = \frac{1}{B+C} \left( Q(t) \right) \left( 1 - \alpha(y) \right) - \alpha(y) \int_{0}^{\pi/2} \alpha(y) \, dy \]

Equilibrium temperature profile:
\[ T^{*}(y) = \frac{1}{B+C} \left( Q(t) \right) \left( 1 - \alpha(y) \right) - \alpha(y) \int_{0}^{\pi/2} \alpha(y) \, dy \]

where \( T^{*} = \frac{1}{B+C} \left( Q(t) \right) \left( 1 - \alpha(y) \right) \) and \( \alpha(y) = \int_{0}^{\pi/2} \alpha(y) \, dy \)

Budyko's Model

Ice Albedo Feedback

Temperature warms
ice melts
albedo decreases
more sunlight absorbed
Temperature warms
Repeat

Why would it stop?


http://www.inenco.org/index_principals.html
Budyko’s Model

Ice Albedo Feedback

What if the albedo is not constant?

Ice Line Assumption: There is a single ice line at \( y = \eta \) between the equator and the pole. The albedo is \( \alpha_1 \) below the ice line and \( \alpha_2 \) above it.

\[
\alpha(y, \eta) = \begin{cases} 
\alpha_1 & y < \eta \\
\alpha_2 & y > \eta 
\end{cases}
\]

Equilibrium condition:

\[
(\beta(\eta)(1 - \alpha(y, \eta)) - (\delta T^*_e(y)) + \Gamma(T - T^*_e(y)) = 0
\]

Equilibrium solution:

\[
T^*_e(y) = \frac{1}{\beta + \Gamma} \left( \beta(\eta)(1 - \alpha_{\eta}) - \delta \right)
\]

where

\[
\beta = \frac{1}{\beta(\eta) \int_0^\eta x(\eta) \, dx + \int_{\eta}^{\infty} x(\eta) \, dx}
\]

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### Budyko’s Model

**Ice Albedo Feedback**

**Summary**

If we artificially hold the ice line at a fixed latitude, then the surface temperature will come to an equilibrium.

However, if the temperature is high, we would expect ice to melt and the ice line to retreat to higher latitudes. If the temperature is low, we would expect ice to form and the ice line to advance to lower latitudes.

*How to model this expectation?*

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**Budyko’s Model**

**Ice Albedo Feedback**

\[
\frac{dT}{dt} = C(T - T_s) - \frac{1}{2} \left( \frac{\alpha(y) \delta(y)}{B + C} - \delta(y) \right) T
\]

Equilibrium:

\[
T_c^*(y) = \frac{1}{B + C} \left( \alpha(y) \delta(y) - \delta(y) \right) - \frac{1}{B + C} C T
\]

Ice line condition:

\[
\frac{1}{B + C} \left( T_c^*(y) + T_c^*(y^*) \right) = T_s = -10
\]

Albedo:

\[
\alpha(y, \eta) = \begin{cases} 
\alpha(y) & \eta = \eta^*, \\
\alpha(y) + \eta & \eta \neq \eta^*
\end{cases}
\]

Ice line condition:

\[
\frac{1}{B + C} \left( T_c^*(y) + T_c^*(y^*) \right) = \frac{1}{B + C} \left( (\alpha(y, \eta) - \alpha(y, \eta^*)) + C T \right) = T_s = -10
\]

where:

\[
\alpha(y) = \frac{\int_{\eta^*}^{\eta^*+} \alpha \, d\eta - \alpha(y, \eta^*) \delta(y) \, \eta^*}{\int_{\eta^*}^{\eta^*+} \delta \, d\eta - \delta(y) \, \eta^*} = 0.47
\]

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**Budyko’s Model**

**Ice Albedo Feedback**

\[
\frac{dT}{dt} = C(T - T_s) - \frac{1}{2} \left( \frac{\alpha(y, \eta) \delta(y) \delta(y^*)}{B + C} - \delta(y) \delta(y^*) \right) T
\]

Equilibrium:

\[
T_c^*(y) = \frac{1}{B + C} \left( \alpha(y, \eta) \delta(y) \delta(y^*) - \delta(y) \delta(y^*) \right) - \frac{1}{B + C} C T
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where:

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**Budyko’s Model**

**Ice Albedo Feedback**

\[
\frac{dT}{dt} = C(T - T_s) - \frac{1}{2} \left( \frac{\alpha(y, \eta) \delta(y) \delta(y^*)}{B + C} - \delta(y) \delta(y^*) \right) T
\]

Equilibrium:

\[
T_c^*(y) = \frac{1}{B + C} \left( \alpha(y, \eta) \delta(y) \delta(y^*) - \delta(y) \delta(y^*) \right) - \frac{1}{B + C} C T
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Ice line condition:

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\frac{1}{B + C} \left( T_c^*(y) + T_c^*(y^*) \right) = \frac{1}{B + C} \left( (\alpha(y, \eta) - \alpha(y, \eta^*)) + C T \right) = T_s = -10
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where:

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\alpha(y) = \frac{\int_{\eta^*}^{\eta^*+} \alpha \, d\eta - \alpha(y, \eta^*) \delta(y) \, \eta^*}{\int_{\eta^*}^{\eta^*+} \delta \, d\eta - \delta(y) \, \eta^*} = 0.47
\]
Budyko’s Model

**Ice Albedo Feedback**

Equilibrium temperature profiles

Interesting Solutions:
- small cap
- large cap
- ice free
- snowball

**Budyko’s Model**

Dynamics of the Ice Line

\[
\frac{\partial T}{\partial t} = Q(T(\eta) - (A + BT) + C(T - T_0))
\]

**Widiasih’s equation:**

\[
\frac{d\eta}{dt} = \alpha(T(\eta) - T_0)
\]

**Widiasih’s Theorem.** For sufficiently small \( \alpha \), the system has an attracting invariant curve given by the graph of a function \( \phi \), \([0,1] \to X\). On this curve, the dynamics are approximated by the equation

\[
\frac{d\eta}{dt} \approx \alpha(T(\eta) - T_0)
\]

**Note:**

- What about the greenhouse effect?
- \( A + BT \) is the outgoing longwave radiation. This term decreases if the greenhouse gases increase.
- We view \( A \) as a parameter.

\[
\frac{d\eta}{dt} = \alpha(T(\eta) - T_0)
\]
Budyko's Model

\[
\frac{dA}{dt} = \alpha(h - A)
\]

isocline \( \alpha(h - A) = 0 \)

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Snowball Earth

Is it possible for Earth to become completely covered in ice? (Snowball Earth)

*Did it ever happen?*


Snowball Earth

"Ice-rafted debris" occurred in ocean sediments near the equator, indicating large equatorial glaciers calving icebergs.

http://www.snowballearth.org/when.html

Snowball Earth

There is evidence that Snowball Earth has occurred, the last time about 600 million years ago.

http://www.snowballearth.org/snow.html

Snowball Earth

Large limestone deposits "cap carbonates" are found immediately above the glacial debris, indicating a rapid warming period following the snowball.

http://www.snowballearth.org/snow.html

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Snowball Earth

**Idea:**
When the Earth is mostly ice-covered, silicate weathering slows down, but volcanic activity stays the same, allowing for a build-up of CO2 in the atmosphere.
When the Earth is mostly ice-free, silicate weathering speeds up, drawing down the CO2 in the atmosphere.

Budyko’s Model

**Budyko-Widiasih Model**

\[
\frac{dA}{dt} = -\varepsilon \eta (\eta - \eta_c)
\]

What if \( \eta \) is a dynamical variable?

Simple equation:

\[
\frac{dA}{dt} = \varepsilon (\eta - \eta_c)
\]

\( 0 < \varepsilon < \varepsilon_c \)

New system:

\[
\frac{dA}{dt} = \varepsilon (\eta - \eta_c)
\]

\[
\frac{dh}{dt} = -\varepsilon \eta (\eta - \eta_c)
\]

Budyko’s Model

**Budyko-Widiasih-Paleocarbon Model**

\[
\frac{dA}{dt} = -\varepsilon \eta (\eta - \eta_c), \quad \frac{dh}{dt} = \varepsilon (\eta - \eta_c)
\]

Stable rest point

What if \( \eta_c \) were here?

Suggested Reading

- Hoffman & Schrag, *Snowball Earth*, SCIENTIFIC AMERICAN, January 2000, 68-75
- K.K. Tung, *Topics in Mathematical Modeling*, PRINCETON UNIVERSITY PRESS, 2007, Chapter 8