An Introduction to Energy Balance Models

Alice Nadeau
(with a lot of slides from Dick McGehee)

University of Minnesota Mathematics of Climate Seminar

September 25, 2018
Conservation of energy

\[ \text{temperature change} \sim \text{energy in} - \text{energy out} \]

\[ \text{energy in} = \text{short wave radiation} - \text{long wave radiation} \]
Conservation of energy

\[
\text{temperature change} \sim \left(\text{energy in} - \text{energy out}\right) \\
\text{short wave radiation} - \text{long wave radiation}
\]

Everything else is detail!
Initial Thoughts

Annual radiation from the Sun := $Q$
Finding $Q$

The energy that is twice as far from the sun is spread over four times the area, making it one-fourth the intensity.

\[
Q = I_{\text{Earth}} \cdot \frac{4 \pi r^2_{\text{Earth}}}{4 \pi r^2_{\text{Sun}}} 
\]

\[
I_{\text{Earth}} = \frac{\text{power flux} \cdot \text{surface area}}{4 \pi r^2_{\text{Earth}}} = \frac{(\sigma T_{\text{Sun}})^4 (4 \pi r^2_{\text{Sun}})}{4 \pi r^2_{\text{Earth}}} \approx 1368 \text{ W m}^{-2}
\]
Finding $Q$

The energy that is twice as far from the sun is spread over four times the area, making it one-fourth the intensity.

$$I_{\text{Earth}} = \frac{\text{power flux} \cdot \text{surface area}}{4\pi r_{\text{Earth}}^2} = \frac{(\sigma T_{\text{Sun}})^4 (4\pi r_{\text{Sun}}^2)}{4\pi r_{\text{Earth}}^2} \approx 1368 \text{ W m}^{-2}$$

$$Q = \frac{I_{\text{Earth}} \cdot \pi r_{\text{Earth}}^2}{4\pi r_{\text{Earth}}^2} \approx 342 \text{ W m}^{-2}$$
Initial Thoughts

Annual radiation from the Sun \( := Q \)

Outgoing radiation \( := \sigma T^4 \)

\[ \rightarrow \text{Stefan-Boltzmann Law} \]
Dynamical Models

Perfect thermally conducting black body:

\[ R \frac{dT}{dt} = Q - \sigma T^4 \]
Dynamical Models

Perfect thermally conducting black body:

\[
R \frac{dT}{dt} = Q - \sigma T^4, \quad T^* = (Q/\sigma)^{1/4}
\]
Dynamical Models

Perfect thermally conducting black body:

\[ R \frac{dT}{dt} = Q - \sigma T^4, \quad T^* = (Q/\sigma)^{1/4} \]

Perfect thermally conducting black body plus albedo:
Albedo
Dynamical Models

Perfect thermally conducting black body:

\[ R \frac{dT}{dt} = Q - \sigma T^4, \quad T^* = (Q/\sigma)^{1/4} \]

Perfect thermally conducting black body plus albedo:

\[ R \frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4, \quad T^* = ((1 - \alpha)Q/\sigma)^{1/4} \]
Dynamical Models for *Surface Temperature*

Convert to surface temperature:

\[ R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT), \quad T^* = \frac{(1 - \alpha)Q - A}{B} \]
Dynamical Models for *Surface Temperature*

Convert to surface temperature:

\[ R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT), \quad T^* = \frac{(1 - \alpha)Q - A}{B} \]

Include latitude dependence:

\[ R \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha) - (A + BT(y, t)), \quad T^*(y) = \frac{(1 - \alpha)Q_s(y) - A}{B} \]
Dynamical Models for *Surface Temperature*

Convert to surface temperature:

\[ R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT), \quad T^* = ((1 - \alpha)Q - A)/B \]

Include latitude dependence:

\[ R \frac{dT}{dt} = Q_s(y)(1 - \alpha) - (A + BT(y, t)), \quad T^*(y) = ((1 - \alpha)Q_s(y) - A)/B \]

Include heat transport:

\[ R \frac{dT}{dt} = Q_s(y)(1 - \alpha) - (A + BT(y, t)) - C \cdot f(T), \quad T^*(y) = ... \]
Budyko vs. Sellers

Mikhail I. Budyko

William D. Sellers
The Budyko Energy Balance Model

\[ R \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha) - (A + BT(y, t)) - C \left( T(y, t) - \overline{T}(t) \right) \]

- **incoming radiation**
- **OLR**
- **heat transport**
- **ann. avg. temp.**

Symmetry assumption:

\[ \text{Equator} = 0 \leq y = \sin(\text{latitude}) \leq 1 = \text{North Pole} \]
Incoming Solar Radiation Distribution: $s(y)$

Dashed: from first principles, Solid: Quadratic approximation
Finding $A$, $B$, and $C$

$A$, $B$, and $C$ are empirical parameters

\[ A = \frac{\text{kcal}}{\text{cm}^2 \text{ month}} \]

\[
R \frac{\partial T}{\partial t} = \underbrace{Qs(y)(1 - \alpha)}_{\text{insolation}} - \underbrace{(A + BT(y, t))}_{\text{albedo}} - \underbrace{C\left(T(y, t) - \frac{\int_0^1 T(y, t)dy}{T(t)}\right)}_{\text{OLR}} - \underbrace{C\left(T(y, t) - \int_0^1 T(y, t)dy\right)}_{\text{heat transport}}
\]

**Incoming Solar Radiation Approximation:**
\[s(y) \approx 1 - 0.238(3y^2 - 1)\]

**Symmetry assumption:**
Equator = 0 \leq y = \sin(\text{latitude}) \leq 1 = \text{North Pole}
Equilibrium Temperature Profile

\[0 = Q_s(y)(1 - \alpha) - (A + B T^*(y)) - C \left( T^*(y) - \overline{T}^* \right)\]
Equilibrium Temperature Profile

$$0 = Qs(y)(1 - \alpha) - (A + BT^*(y)) - C\left(T^*(y) - \overline{T}^*\right)$$

Integrate to find $\overline{T}^*$:

$$0 = \int_0^1 \left[ Qs(y)(1 - \alpha) - (A + BT^*(y)) - C\left(T^*(y) - \overline{T}^*\right) \right] dy$$

$$= Q\int_0^1 s(y)dy - Q\int_0^1 s(y)\alpha dy - A\int_0^1 dy - B\int_0^1 T^*(y) dy$$

$$- C\int_0^1 T^*(y)dy + C\int_0^1 \overline{T}^* dy$$

$$= Q(1 - \alpha) - (A + B\overline{T}^*)$$

$$\Rightarrow \text{Equilibrium Global Mean Temperature: } \overline{T}^* = \frac{1}{B}(Q(1 - \alpha) - A)$$
Equilibrium Temperature Profile

\[ 0 = Q_s(y)(1 - \alpha) - (A + B\bar{T}^*(y)) - C\left(\bar{T}^*(y) - \bar{T}^*\right) \]

\[ \bar{T}^* = \frac{1}{B}(Q(1 - \bar{\alpha}) - A) \]
Equilibrium Temperature Profile

\[ 0 = Q_s(y)(1 - \alpha) - (A + B\overline{T}^*(y)) - C\left(\overline{T}^*(y) - \overline{T}^*\right) \]

\[ \overline{T}^* = \frac{1}{B}(Q(1 - \overline{\alpha}) - A) \]

Plug in \( \overline{T}^* \) and solve for \( T^*(y) \):

\[ T^*(y) = \frac{1}{B + C}(Q_s(y)(1 - \alpha) - A + C\overline{T}^*) \]
\[ T^*(y) = \frac{1}{B + C} \left( Q_s(y)(1 - \alpha) - A + C\overline{T^*}\right) \]

\[ \alpha = 0.32 \]
\[ \alpha = 0.62 \]
\[ C = 3.04 \]

From McGehee, Climate Seminar Sept. 19, 2017
Ice-Albedo Feedback

- Melting of sea ice
- Lowered albedo
- Increase in absorbed sunlight

Conservation of Energy
Ice-Albedo Feedback
Dynamic Ice Line
Studying Climate
Further Reading
Non-uniform Albedo

\[ R \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + BT(y, t)) - C \left( T - \overline{T}^* \right) \]

albedo depends on latitude
Non-uniform Albedo

\[ R \frac{\partial T}{\partial t} = Q_s(y) (1 - \alpha(y, \eta)) - (A + B T(y, t)) - C (T - \overline{T}^*) \]

albedo depends on latitude

**Ice Line Assumption:** There is one ice line, \( \eta \), in the northern hemisphere north of which there is always ice.

\[ \alpha(y, \eta) = \begin{cases} 
\alpha_1 & 0 \leq y < \eta \\
\alpha_2 & \eta < y \leq 1 
\end{cases}, \quad \alpha_1 < \alpha_2 \]
Equilibrium Temperature Profile depends on the Ice Line

\[ T^*_\eta(y) = \frac{1}{B + C} \left( Qs(y)(1 - \alpha) - A + C \bar{T}^* \right) \]

\[ \bar{T}^*_\eta = \frac{1}{B} (Q(1 - \bar{\alpha}(\eta)) - A) \]

where

\[ \bar{\alpha}(\eta) = \int_0^1 s(y)\alpha(y, \eta)dy = \alpha_1 \int_0^\eta s(y)dy + \alpha_2 \int_\eta^1 s(y)dy \]
Equilibrium Temperature Profile depends on the Ice Line

From McGehee, Climate Seminar Sept. 19, 2017
Experts only:

**Theorem (Widiasih)**

Let $X$ be the space of functions where $T$ lives and

$$L : X \rightarrow X; \quad LT := C\bar{T} - (B + C)T.$$ 

If $f(y) = Qs(y)(1 - \alpha(y, \eta)) - A$, then Budyko's equation can be written as a linear vector field on $X$:

$$R \frac{dT}{dt} = f + LT.$$ 

Furthermore, the operator $L$ has only point spectrum, with all eigenvalues negative. Therefore all solutions are stable.
Dynamics of $T$

Experts only:

Theorem (Widiasih)

Let $X$ be the space of functions where $T$ lives and

$$L : X \rightarrow X; \quad LT := C \bar{T} - (B + C)T.$$  

If $f(y) = Qs(y)(1 - \alpha(y, \eta)) - A$, then Budyko's equation can be written as a linear vector field on $X$:

$$R \frac{dT}{dt} = f + LT.$$  

Furthermore, the operator $L$ has only point spectrum, with all eigenvalues negative. Therefore all solutions are stable.

Everyone: For each fixed ice line $\eta$, there is a globally stable equilibrium solution for Budyko's equation.
Something seems wrong...
Something seems wrong...

Intuition:

- High temperature $\Rightarrow$ ice melts $\Rightarrow$ ice line moves north

- Low temperature $\Rightarrow$ ice forms $\Rightarrow$ ice line moves south
Something seems wrong...

Intuition:

- High temperature $\Rightarrow$ ice melts $\Rightarrow$ ice line moves north
- Low temperature $\Rightarrow$ ice forms $\Rightarrow$ ice line moves south

How do we model our intuitions?
**Ice Formation Assumption:** Permanent ice forms if the annual average temperature is below $T_c = -10 \, ^\circ C$ and melts if the annual average temperature is above $T_c$.
Dynamic Ice Line

**Ice Formation Assumption:** Permanent ice forms if the annual average temperature is below $T_c = -10 \, ^\circ C$ and melts if the annual average temperature is above $T_c$

\[
\frac{d\eta}{dt} = \epsilon(T^*_\eta(\eta) - T_c)
\]
Dynamics of the Ice Line

\[ R \frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y, \eta)) - (A + BT(y, t)) - C \left( T - \overline{T}_\eta^* \right), \quad \frac{d\eta}{dt} = \epsilon \left( T^*_\eta(\eta) - T_c \right) \]

Experts only:

**Theorem (Widiasih’s Theorem)**

*For sufficiently small \( \epsilon \), the system has an attracting invariant curve given by the graph of a function \( \Phi_\epsilon : [0, 1] \rightarrow X \). On this curve, the dynamics are approximated by the equation*

\[ \frac{d\eta}{dt} = \epsilon \left( T^*_\eta(\eta) - T_c \right). \]

Dynamics of the Ice Line

\[ R \frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y, \eta)) - (A + BT(y, t)) - C \left( T - T_\eta^* \right), \quad \frac{d\eta}{dt} = \epsilon(T_\eta^*(\eta) - T_c) \]

Everyone: \( h(\eta) = T_\eta^*(\eta) - T_c \)

From McGehee, Climate Seminar Sept. 19, 2017
The Budyko–Widiasih Model

\[
R \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + B T(y, t)) - C \left( T - \overline{T}_{\eta}^* \right), \quad \frac{d\eta}{dt} = \epsilon (T_{\eta}^*(\eta) - T_c)
\]

From McGehee, Climate Seminar Sept. 19, 2017
Greenhouse Gasses in the Budyko–Widiasih Model

\[ R \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + B T(y, t)) - C \left( T - T^* \right) \]

outgoing long wave radiation

\[ \frac{d\eta}{dt} = \epsilon h(\eta, A) \]

The parameter \( A \) is the **greenhouse gas parameter.**
Bifurcation Diagram for $A$

\[ \frac{d\eta}{dt} = \epsilon h(\eta, A) \]
Current Earth

Current value of A corresponds to small, year-round ice caps at the poles

Photo: NASA
Bifurcation Diagram for $A$

\[ \frac{d\eta}{dt} = \epsilon h(\eta, A) \]
Small values of $A$ correspond to an Earth that is too warm for year-round ice caps.
Bifurcation Diagram for $A$

\[ \frac{d\eta}{dt} = \epsilon h(\eta, A) \]
Past Earth?

Large values of $A$ correspond to a completely ice covered Earth

Photo: BBC
Evidence for Snowball Earth

Hoffman & Schrag, Snowball Earth, Scientific American, January 2000, 68-75
Everyone:


MeGehee and Widiasih. (2014) “A quadratic approximation to Budyko’s ice-albedo feedback model with ice line dynamics.”

Walsh (2016) “Periodic orbits for a discontinuous vector field arising from a conceptual model of glacial cycles.”


Experts:


MeGehee and Widiasih. (2014) “A quadratic approximation to Budyko’s ice-albedo feedback model with ice line dynamics.”

Walsh (2016) “Periodic orbits for a discontinuous vector field arising from a conceptual model of glacial cycles.”

Thank you!