Climate, Non-Smooth Dynamics, and Conley Theory

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Overview

1. Welander’s Ocean Box Model
2. Filippov Systems
3. Filippov and Flows
4. Multiflows
5. Nearby Smooth Systems
Atlantic Meridional Overturning Circulation

The Atlantic Meridional Overturning Circulation is a component of ocean currents that moves heat and water around the Earth. Its behavior has a strong influence on our climate.

Image: [21]
There is strong evidence that AMOC has changed convective strength in the past.
Welander’s goal: Prove these changes could be internally driven, instead of relying on outside forcing [20].
Welander’s Model

Ocean circulation box model: Planar system, salt (S) and temperature (T) are dynamic variables.

Welander’s goal: Show internally driven ocean convection strength oscillations, instead of relying on outside forcing.

Figure: Deep Ocean and Shallow Ocean [20]
Welander’s Model

\[ \dot{T} = k_T(T_A - T) - k(\rho)T \]
\[ \dot{S} = k_S(S_A - S) - k(\rho)S \]
\[ \rho = -\alpha T + \gamma S \]

Smooth Version:

\[ k(\rho) = \frac{1}{\pi} \tan^{-1}(\frac{\rho - \epsilon}{a}) + \frac{1}{2} \]

Nonsmooth Version:

\[ k(\rho) = \begin{cases} 
  k_1, & \rho > \epsilon \\
  0, & \rho < \epsilon 
\end{cases} \]

\[ \Sigma: \text{Line } \rho = \epsilon \]

Figure: Deep Ocean and Shallow Ocean [20]
When a smooth $k$ is used, the system can be analyzed using traditional methods.

Welander uses Poincare-Bendixson to find a periodic orbit and get a proof-of-concept for his convective oscillation idea.
When a non-smooth $k$ is used, the system cannot be analyzed using traditional methods.

It is clear that this model was Welander’s original motivation, however.

**Figure**: Non-Smooth Version of Welander’s model. The red dots are equilibria on opposite sides of the switching boundary. [10]
Welander’s non-smooth model is an example of a Filippov System.

\[ \dot{x} \in F(x) = \begin{cases} f_1(x), & x \in G_1 \\ f_2(x), & x \in G_2 \\ \{\alpha f_2(x) + (1 - \alpha)f_1(x) : \alpha \in [0, 1]\} & x \in \Sigma \end{cases} \]

**Figure:** A planar Filippov system with $\mathbb{R}^2$ split into two regions.
Filippov Systems and Climate

Many other climate models are also Filippov Systems:

- Earth’s surface albedo [4][6]
- Ecological Decision Making [17]
- Socio-Economic Decision Making [19]
- Layers in the Atmosphere? (Yorkinoy Shermatova)
Periodic Orbit in the Non-Smooth Welander Model

Figure: Periodic orbit in the non-smooth model.

Julie Leifeld used methods from Filippov’s book to prove that the periodic orbit exists in the non-smooth model. [10]
Periodic Orbit in the Non-Smooth Welander Model

Figure: Periodic orbit in the non-smooth model.

Problems:

- The methods Dr. Leifeld used are very ad-hoc (cannot be easily applied to other models).
- There are few known techniques for analysis in very high dimensions.
Conley Index Theory for Filippov Systems?

- Conley Index Theory is a powerful, (relatively) easy to use tool that gives robust topological information about a system.
- It works in arbitrarily high dimensions.
- It seems to work for Filippov systems.

*Figure: Charles Cameron Conley*
Conley Theory works with Welander

Figure: In the Welander Model, we can choose $N$ to be homeomorphic to an annulus, and it has an empty exit set $L$, so the Conley Index is of a circle and a disjoint point.

- A compact isolating neighborhood $N$ (no invariant points on boundary).
- The exit set $L \subset \partial N$
- The Conley Index is the homology sequence associated to the pair $(N/L, [L])$. 
Can Conley Theory be Extended to Filippov Systems?

- For all of our well-understood dynamics, Conley Theory seems to work, but we cannot prove (yet) that it works in general.
- Key Issue: Conley Theory requires a flow, but Filippov systems do not give rise to flows.
- Richard McGehee’s Solution: the multiflow.
A flow is a continuous map \( \varphi : \mathbb{R} \times X \rightarrow X \) satisfying the group properties

- \( \varphi(0, x) = x \)
- \( \varphi(s, \varphi(t, x)) = \varphi(s + t, x) \)

The flow relates to the differential equation

\[
\dot{x} = f(x)
\]

by letting \( \varphi(t, x_0) \) correspond to the solution \( x(t) \) with the initial condition \( x(0) = x_0 \).
Why Can’t Filippov Systems give Flows?

Filippov systems have:
- Intersecting trajectories
- Non-unique solutions

This prevents Filippov systems from being flows:
- No group action
- Cannot be a map

This is where multiflows come in, but let’s look at the behavior of Filippov systems first a bit.
Behavior Near Splitting Boundary

- **Figure**: Crossing Region
- **Figure**: Attracting Region
- **Figure**: Repelling Region
Intersecting Trajectories

Figure: Intersecting Trajectories in a simple Filippov System

Cannot obey group properties:

$$\phi_t(\phi_{-t}(x)) = \phi_{t-t}(x) \neq \phi_0(x) = x$$
Intersecting Trajectories

Figure: Intersecting Trajectories in a Filippov System

Solution: Monoid Action (Semiflow)
Multiple Solutions

Figure: Four different solutions of a Filippov system

\[ (x, y) \in H(x, y) := \begin{cases} 
(1, x), & y > 0 \\
(1, \beta) : \beta \in [-x, x], & y = 0 \\
(1, -x), & y < 0 
\end{cases} \]
A multiflow is an object that is intended to generalize the concept of flows to Filippov systems.

Before we define multiflows, we need some background.
Relations

A **relation** on a topological space $X$ is a subset of $X \times X$.

If $F$ and $G$ are both relations on $X$, then we can define the composition:

$$F \circ G = \{(x, z) \in X \times X : \exists y \in X \text{ s.t. } (x, y) \in G, (y, z) \in F\}$$
The Closed Graph Theorem

Let $X$ be a topological space and let $Y$ be a Hausdorff space.

$f : X \to Y$ is continuous

$\downarrow$

The graph of $f$ is closed
The Closed Graph Theorem

Let $X$ be a topological space and let $Y$ be a compact Hausdorff space.

$f : X \rightarrow Y$ is continuous

$\uparrow$

The graph of $f$ is closed
The graph of a flow $\phi$ on a compact set $X$ is a closed subset of $\mathbb{R} \times X \times X$ such that for each $t \in \mathbb{R}$, $\phi^t$ contains exactly one pair $(x, y) \in X \times X$ for each $x \in X$ and the group properties hold:

- $\phi^0 = \{(x, x) : x \in X\}$
- $\phi^{t+s} = \phi^t \circ \phi^s$

Where $\phi^t := \{(x, y) \in X \times X : (t, x, y) \in \phi\}$
Can we modify flows to fit Filippov Systems?

The graph of a flow $\phi$ on a compact set $X$ is a closed subset of $\mathbb{R}^+ \times X \times X$ such that for each $t \in \mathbb{R}$, $\phi^t$ contains exactly one pair $(x, y) \in X \times X$ for each $x \in X$ and the group monoid properties hold:

- $\phi^0 = \{(x, x) : x \in X\}$
- $\phi^{t+s} = \phi^t \circ \phi^s$

Where $\phi^t := \{(x, y) \in X \times X : (t, x, y) \in \phi\}$
A **multiflow** $\Phi$ on a compact space $X$ is a closed subset of $\mathbb{R}^+ \times X \times X$ satisfying the monoid properties:

- $\Phi^0 = \{(x, x) : x \in X\}$
- $\Phi^{t+s} = \phi^t \circ \phi^s$

Where $\Phi^t := \{(x, y) \in X \times X : (t, x, y) \in \Phi\}$
**Theorem:** Let $\dot{x} \in F(x)$ be a Filippov system on an open domain $G \subset \mathbb{R}^n$, and let $K \subset G$ be compact. Let $\Phi$ be the set of all points

$$\{(T, a, b) \in \mathbb{R}^+ \times K \times K\}$$

such that there exists a solution $x : [0, T] \to K$ satisfying $x(0) = a$ and $x(T) = b$.

Then the set $\Phi$ is a multiflow over $K$.

**Figure:** Once solutions leave $K$, they are no longer included in $\Phi$. 
Nearby Smooth Systems and Conley Theory

Conley Theory for multiflows: still in progress.

Since Conley Theory is robust under perturbation, if we can extend it we can get information about nearby smooth systems.
Researchers are often interested in how well a non-smooth system approximates a limiting smooth system, much like in the Welander model:

**Smooth Version:**

\[
k(\rho) = \frac{1}{\pi} \tan^{-1}\left(\frac{\rho - \epsilon}{a}\right) + \frac{1}{2}
\]

**Nonsmooth Version:**

\[
k(\rho) = \begin{cases} 
  k_1 & \rho > \epsilon \\
  0 & \rho < \epsilon 
\end{cases}
\]

**Figure:** As $\alpha \to \infty$, $\tanh \alpha x$ limits to a piecewise function.
Filippov Systems are actually *differential inclusions*

\[ \dot{x} \in F(x) \]

where \( F \) is a set-valued map.

A solution is an absolutely continuous function satisfying

\[ \frac{d}{dt} x(t) \in F(x(t)) \]

almost everywhere on some interval \( I \in \mathbb{R} \).
Jeffrey’s Objection

Mike Jeffrey showed that infinitely many smooth systems can limit to the same piecewise system, but their behavior near the switching boundary can be different. [7]

Figure: The function $\tanh(\alpha x) + 2$ and the same function modified by a smooth mollifier.

Figure: Convex combination to produce $F(x)$ from discontinuous $f$ above.
Use Hausdorff Metric to Define "Nearby"

The convex combination method is popular, but Filippov worked in more generality. Using the Hausdorff metric we can define "nearby" more appropriately for this setting.
References I


References II


