Convergence and Equilibrium for Stochastic Models of Ecological Disturbances

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California wildfires, 2019

A firefighting helicopter flies over the Getty Fire as it burns in the hills west of the 405 freeway.

Gene Blevins/Reuters
The Kincade Fire approaches a herd of alpacas in Sonoma County.

Ethan Swope/AP
How can one model the carbon content of an ecosystem with randomly occurring disturbances of random severity?
Today’s plan

1. Discrete time and state introduction

2. Continuous time and state
   Semistochastic model
Questions to keep in mind

1. What information can we extract from equilibrium distributions?

2. Why do convergence rates matter?
Markov chains

1. $\mathcal{X}$ : State space
2. $\mu_0$ : Initial distribution
3. $Q$ : Transition matrix
Two-state Markov chain

1. $\mathcal{X} : \{\text{Fire, No Fire}\}$
2. $\mu_0 : \text{Is there a fire now?}$
3. $Q : \text{Environment, beliefs}$
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Evolved distributions

- Probability vectors

\[ \mu_n = \begin{pmatrix} \text{Prob. of fire on } n^{th} \text{ day} \\ \text{Prob. of no fire on } n^{th} \text{ day} \end{pmatrix}^T \]

- Initial probability vector: \( \mu_0 \)

\[ \mu_n = \mu_0 \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{50} & \frac{49}{50} \end{pmatrix}^n = \mu_0 Q^n \]
Equilibrium distributions

- Limiting:
\[
\lim_{n \to \infty} \mu_n = \pi \quad \text{for any } \mu_0
\]

- Invariant:
\[
\mu_0 = \pi \Rightarrow \mu_n = \pi \quad \text{for all } n > 0
\]

Note: These two characterizations are not always equivalent!
An invariant approach

\[ \pi = \pi Q \]

\[ = \pi \begin{pmatrix} \frac{9}{10} & \frac{1}{10} \\ \frac{1}{50} & \frac{49}{50} \end{pmatrix} \]

\( \pi \) is a *left*-eigenvector of the transition matrix, \( Q \), with eigenvalue 1.

\[ \pi = \left( \frac{1}{6}, \frac{5}{6} \right) \]
Question:
What information can we extract from this equilibrium distribution?

Answer:
In the long run, there will be fires on 1 out of 6 days.
Realization of Markov chain

Fire
No Fire

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The eigenvalues of $Q$ satisfy $\lambda_1 = 1$ and $|\lambda_2| = 1 - \beta$ with 
\[
\beta = \left| 1 - \frac{1}{10} - \frac{1}{50} \right| = \frac{6}{50}.
\]

$$
Q = \begin{pmatrix}
\frac{9}{10} & 1 \\
\frac{1}{10} & \frac{1}{10} \\
\frac{1}{50} & 49 \\
\frac{1}{50} & \frac{50}{50}
\end{pmatrix}
$$

The other eigenvalue is related to the rate at which an arbitrary initial distribution, $\mu_0$, converges to $\pi$. One can show

$$
d_{TV}(\mu_n, \pi) \leq (1 - \beta)^n.
$$
Given probability distributions, $\mu$ and $\nu$: 

$$d_{TV}(\mu, \nu) = \sup_{A} |\mu(A) - \nu(A)|$$

$$d_{TV}(\mu, \nu) = \sup_{0 \leq f \leq 1} |\mathbb{E}_\mu(f) - \mathbb{E}_\nu(f)|$$
Total variation distance
Total variation distance

\[ d_{TV}(\pi, \mu_t) \]

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Total variation distance
Question:
Why do convergence rates matter?

Answer:
The “long run” may be a long time coming.

Markov chains are “memoryless”, but need time to forget.
Evolution of probabilities

Fire Simulation

- Fires
- No Fires

Likelihood vs. Number of Days in Future

0 20 40 60 80 100

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FAST:

SLOW:

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Convergence and Equilibrium for Stochastic Models of Ecological Disturbances
How can one model the carbon content of an ecosystem and account for randomly occurring disturbances of random severity?
Semistochastic model for the carbon content of an ecosystem

Model design:

- Carbon content increases deterministically between disturbances.
- Fires occur at random times and release carbon.
- Severity of fires is random.
Semistochastic model

Time

Carbon

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Components of semistochastic model

**Growth Rate** – Deterministic evolution

\[
\frac{dx}{dt} = g(x), \quad x(t) = \phi^t(x_0)
\]

**Disturbance Rate** – Probability per unit time

\[\Lambda(x)\]

**Disturbance Kernel** – Severity of disturbances

\[P(x, A)\]
Components of semistochastic model

**Growth Rate** – Deterministic evolution

\[
\frac{dx}{dt} = g(x), \quad x(t) = \phi^t(x_0)
\]

**Disturbance Rate** – Probability per unit time

\[\Lambda(x)\]

**Disturbance Kernel** – Severity of disturbances

\[P(x, A)\]

This state dependence is important!!
Disturbance kernel

\[ x \rightarrow P(x, A) \]

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Stochastic flow-kick model

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Distribution of $X_t$

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Evolution operator

- Initial Distribution:
  \[ \mu_0 \]

- Evolved Distributions:
  \[ \mu_0 \rightarrow \mu_t =: \mu_0 \mathcal{U}^t \]
Convergence and Equilibrium for Stochastic Models of Ecological Disturbances
A direct approach: Differential forward equation for time-dependent density

For absolutely continuous distributions and disturbance kernel,

\[ \frac{d\mu_t}{dx} = \rho(x, t), \quad P(x, dy) = p(x, y)dy \]

\[ \partial_t \rho(x, t) = -\partial_x (g(x)\rho(x, t)) - \Lambda(x)\rho(x, t) + \int p(y, x)\Lambda(y)\rho(y, t)dy. \]
A direct approach: Differential forward equation for time-dependent density

For absolutely continuous distributions and disturbance kernel,

\[ \frac{d\mu_t}{dx} = \rho(x, t), \quad P(x, dy) = p(x, y) dy \]

\[ \partial_t \rho(x, t) = -\partial_x (g(x)\rho(x, t)) - \Lambda(x)\rho(x, t) + \int p(y, x)\Lambda(y)\rho(y, t) dy. \]

Deterministic evolution
A direct approach: Differential forward equation for time-dependent density

For absolutely continuous distributions and jump kernel,

\[ \frac{d\mu_t}{dx} = \rho(x, t), \quad P(x, dy) = p(x, y)dy \]

\[ \partial_t \rho(x, t) = - \partial_x (g(x)\rho(x, t)) - \Lambda(x)\rho(x, t) + \int p(y, x)\Lambda(y)\rho(y, t) \, dy. \]

Disturbance occurrence
A direct approach: Differential forward equation for time-dependent density

For absolutely continuous distributions and jump kernel,

\[ \frac{d\mu_t}{dx} = \rho(x, t), \quad P(x, dy) = p(x, y)dy \]

\[ \partial_t \rho(x, t) = -\partial_x (g(x)\rho(x, t)) - \Lambda(x)\rho(x, t) + \int p(y, x)\Lambda(y)\rho(y, t) dy. \]

Disturbance severity
Fundamental questions

- Does there exist a distribution $\pi$ on $\mathcal{X}$ with
  \[ d_{TV}(\mu_t, \pi) \to 0 \text{ as } t \to \infty \]?

- Given $\delta > 0$, how large must $t$ be so that
  \[ d_{TV}(\mu_t, \pi) < \delta \]?
Evolved distributions

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Evolved distributions

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Evolved distributions

$X_t$

dTV($\pi,\mu_t$)

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Evolved distributions

\[ \pi, \mu_t, d_{TV}(\pi, \mu_t) \]
Question:
What information can we extract from this equilibrium distribution?

Answer:
We can compute the fraction of time (in the long run) that the process $X_t$ spends in any measurable subset of our state space.
Theorem (B.)

Under appropriate assumptions, \( X_t \) is uniformly ergodic with a unique stationary distribution, \( \pi \), and

\[
\text{d}_{TV}(\mu_t, \pi) \leq (1 - \beta)^t
\]

for any initial distribution \( \mu_0 \).

Note: The value of \( \beta \) is explicitly constructed.
How is $\beta$ computed?

1. Discretize the process (in time).
2. Develop minorization for the discretization.
3. Deduce bounds for the original continuous-time process.
Discrete time transition kernel

\[ \mathcal{U}^{\Delta t}(x, A) \]
Uniform minorization

Ingredients:

1. Probability measure $\eta$ on $\mathcal{X}$
2. $\beta > 0$

With

$$U^{\Delta t}(x, A) \geq \beta \eta(A)$$

for any measurable set $A$ and all $x \in \mathcal{X}$. 
Let $f : \mathcal{X} \mapsto \mathbb{R}$ be an observable, then

$$\langle \mu_{\Delta t}, f \rangle = \langle \mu_0 \mathcal{U}^{\Delta t}, f \rangle = \langle \mu_0, \mathcal{U}^{\Delta t}f \rangle$$

with

$$[\mathcal{U}^{\Delta t}f](x) := \mathbb{E}[f(X_{\Delta t}) | X_0 = x]$$

and

$$\langle \mu_0, \mathcal{U}^{\Delta t}f \rangle = \int [\mathcal{U}^{\Delta t}f](x)d\mu_0(x)$$
The minorization condition

\[ U^{\Delta t}(x, A) \geq \beta \eta(A) \]

is equivalent to requiring for any nonnegative observable \( f \), and for all \( x \in A \),

\[ [U^{\Delta t}f](x) \geq \beta \int f(y) d\eta(y) . \]
To control the discrete-time evolution operator, $U^{\Delta t}$, we can study the infinitesimal generator $L$ of $U^t$.

$L$ acts on observables, $f$, according to

$$[Lf](x) = \lim_{t \to 0} \frac{U^t f(x) - f(x)}{t}.$$ 

In our case,

$$[Lf](x) = f'(x)g(x) + \Lambda(x) \int P(x, dy)[f(y) - f(x)].$$
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Uniform minorization for compact one-dimensional state spaces

1. \[
\frac{d\eta}{dx} = \frac{\Delta t - \psi(0,x)}{C_1} \mathbb{1}\{0 \leq x \leq \phi^{\Delta t}(0)\}
\]

2. \[
\beta_{\Delta t} = e^{-\lambda \Delta t} C_2 \int_0^{\phi^{\Delta t}(0)} \left[ \Delta t - \psi(0, x) \right] dx
\]

with \(\psi\) defined by

\[
x_1 = \phi^t(x_0) \iff t = \psi(x_0, x_1)
\]

and

\[
\lambda \geq \Lambda(x) \text{ for all } x \in \mathcal{X}
\]
Plots of $(1 - \beta_{\Delta t})^{[t/\Delta t]}$ vs. $t$ for various $\Delta t$
Question: Why do convergence rates matter?

Answers:
- Determine how long before “long-run” averages are realized.
- Provide guidance for numerical methods of approximating stationary distributions.
- Relevant for sub-sampling techniques used with Monte Carlo methods in likelihood-based inference.
Applications of semistochastic/piecewise-deterministic models

- Ecological disturbances
- Precipitation models
- Growth-fragmentation processes
- Human behaviour
- Viral reproduction


