An Introduction to Planetary Energy Balance

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MCRN
Mathematics and Climate Research Network

Energy Balance

Conservation of Energy

temperature change ~ energy in – energy out

energy in: short wave energy from the Sun
energy out: long wave energy from the Earth

Everything else is detail.

Energy Balance

Stefan-Boltzmann Law

$$F = \sigma T^4$$

F: power flux (W/m^2)
T: temperature (K)
σ: Stefan-Boltzmann constant

$\sigma \approx 5.67 \times 10^{-8}$ W/m$^2K^4$

Reasonable approximation:
Every body in the solar system radiates energy according to this law.

Energy Balance

Stefan-Boltzmann Law

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F: power flux (W/m^2)
T: temperature (K)
σ: Stefan-Boltzmann constant

$\sigma \approx 5.67 \times 10^{-8}$ W/m$^2K^4$

Example

surface temperature of the Sun: 5780K
power flux: $5.67 \times 10^{-8} \times (5780)^4 =$
6.33×10^7 W/m^2

total solar power output: $6.33 \times 10^7 \times 4\pi(r_S)^2$,
where r_S = radius of the sun = 6.96×10^8 m
total solar output: 3.85×10^{26} W

Energy Balance

Insolation

Solar flux at a distance r from the sun:

$$F = \frac{6.33\times 10^7\, 4\pi r_S^2}{4\pi r^2} = 6.33\times 10^7 \left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

$r_S = 6.96 \times 10^8$ m
$r = 1.5 \times 10^{11}$ m

$\boxed{F = 1368 \text{ W/m}^2}$ ← **solar flux at Earth's orbit**

Power intercepted by the Earth: $F \times \pi r_E^2$ W

Earth's surface area: $4\pi r_E^2$ m^2

Average surface flux: $\dfrac{F \times \pi r_E^2}{4\pi r_E^2} = \dfrac{F}{4} = \boxed{342 \text{ W/m}^2}$

Energy Balance

Insolation

Global Average Insolation
(Incoming solar radiation)

intercepted flux: $F = 1368$ W/m^2
Earth cross-section: πr_E^2
surface area: $4\pi r_E^2$
average flux: 1368/4 = **342 W/m² = Q**

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = \left(342/5.67\times 10^{-8}\right)^{1/4}$$

$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

← **stable equilibrium**

Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

R: **heat capacity**

Energy Balance

Goldilocks Zone

Solar flux at a distance r from the Sun:

$$F = \frac{6.33\times10^7\, 4\pi r_S^2}{4\pi r^2} = 6.33\times10^7\left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

$$r_S = 6.96\times10^8 \text{ m}$$

$$F = \frac{3.07\times10^{25}}{r^2} \text{ W/m}^2$$

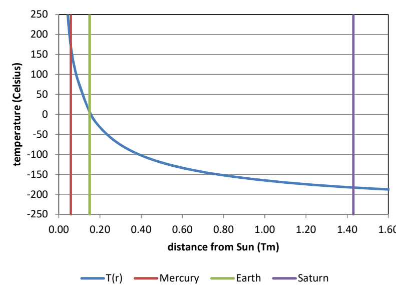
Average surface flux: $\dfrac{3.07\times10^{25}}{4r^2} = \dfrac{7.67\times10^{24}}{r^2}$ W/m^2

Black body temperature: $\sigma T^4 = \dfrac{7.67\times10^{24}}{r^2}$ W/m^2

$$T = \left(\frac{7.67\times10^{24}}{\sigma r^2}\right)^{1/4} = \frac{1.078\times10^8}{\sqrt{r}}$$

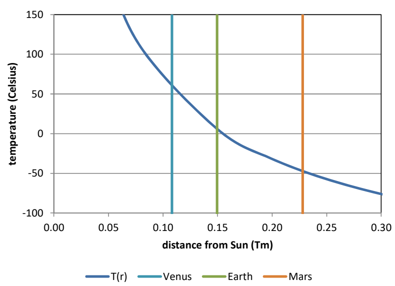
Energy Balance

Goldilocks Zone

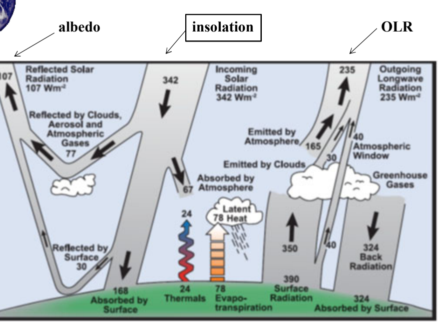


Energy Balance

Goldilocks Zone



Energy Balance



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Energy Balance

Insolation

Global Average Insolation
(Incoming solar radiation)

intercepted flux: $F = 1368$ W/m^2
Earth cross-section: πr_E^2
surface area: $4\pi r_E^2$
average flux: $1368/4 = \mathbf{342\ W/m^2} = Q$

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = (Q/\sigma)^{1/4} = \left(342/5.67\times10^{-8}\right)^{1/4}$$

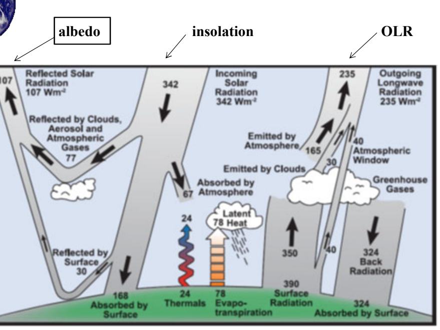
$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity (→ R) **stable equilibrium** (→ 279K)

Energy Balance



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Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.
For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot F/\sigma)^{1/4} = (0.7 \cdot 342/5.67\times10^{-8})^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

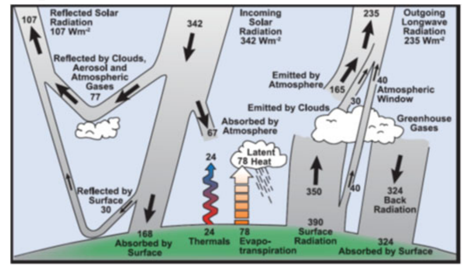
Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

Energy Balance

albedo **insolation** **OLR**



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Energy Balance

OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$A = 202\ \text{W/m}^2$

$B = 1.90\ \text{W/m}^2\text{K}$

Dynamics

Kelvin $$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$ **photosphere temperature**

becomes

Celsius $$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$ **global mean surface temperature**

Energy Balance

OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$

Important:
$A+BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

different

Dynamics

Kelvin $$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$ **photosphere temperature**

becomes

Celsius $$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$ **global mean surface temperature**

Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT_{eq} = 0$

$$T_{eq} = \frac{Q(1-\alpha) - A}{B}$$ Stable, since $B > 0$.

Ice-free planet: $\alpha = 0.32$, $T_{eq} = 16$ °C
Snowball planet: $\alpha = 0.62$, $T_{eq} = -38$ °C

No glacier would form on an ice-free Earth.
No glacier would melt on a snowball Earth.

Easy question:
Why do we have ice caps?

Hard question:
If Earth was ever a snowball, how did we get out?

Energy Balance

Latitude Dependence

Make T depend on $y = \sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

insolation distribution

Q = global annual average insolation = 342W/m^2

$s(y)$ = distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

One can show that

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

β = obliquity = 23.4°

Chylek and Coakley's quadratic approximation:

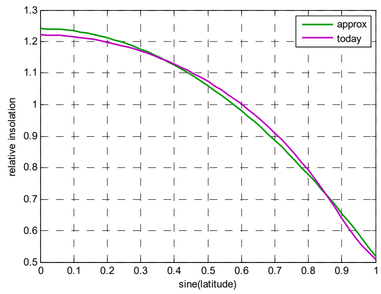
$$s(y) \approx 1 - 0.241(3y^2 - 1)$$

Energy Balance

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.4°



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Energy Balance

Latitude Dependence

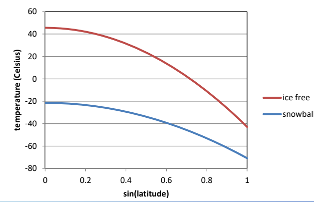
$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Note that y is just a parameter.

Equilibrium Temperature Profile

$$T_{eq}(y) = \frac{Qs(y)(1-\alpha) - A}{B}$$

$\alpha = 0.32$: ice free
$\alpha = 0.62$: snowball



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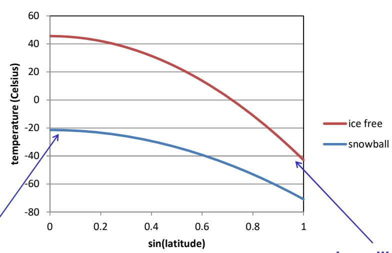
Energy Balance

Latitude Dependence



ice won't melt (no exit from snowball)

ice will form (icecap)

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Energy Balance

What's Missing?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

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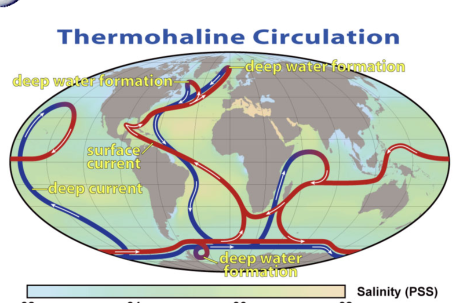
Energy Balance

What's Missing?



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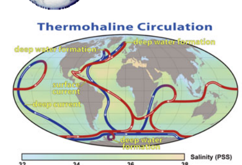
Energy Balance

What's Missing?



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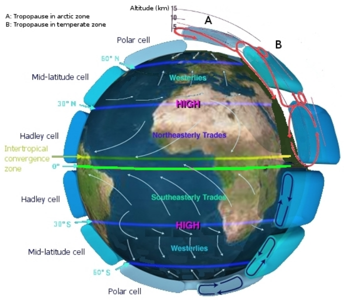
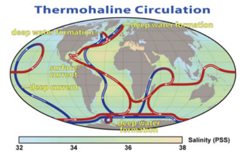
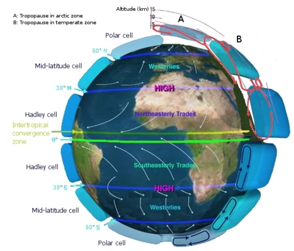


Energy Balance

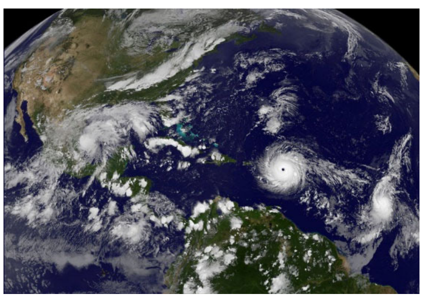
What's Missing?





Energy Balance

What's Missing?

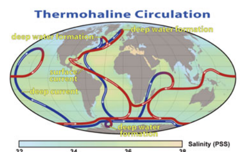


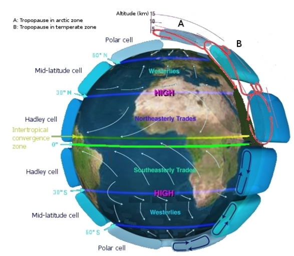
https://www.nytimes.com/2017/09/04/us/hurricane-irma.html

Energy Balance

What's Missing?

Weather!





Energy Balance

Budyko's Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

Weather

global mean temperature $\bar{T}(t) = \int_0^1 T(y,t)\,dy$

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Budyko's equation as a dynamical system:
T lives in a function space (temperature as a function of latitude).

Budyko's Model

Why y?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A+BT(y,t)) + C(\bar{T}(t) - T(y,t))$$

global mean temperature: $\bar{T}(t) = \int_0^1 T(y,t)\,dt$

Why do we use y = sine(latitude) instead of just latitude?

Budyko’s Model

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t)) + C(\overline{T}(t) - T(y,t))$$

global mean temperature $\overline{T}(t) = \int_0^1 T(y,t)dt$

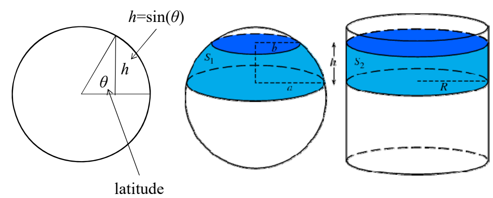
Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.

Budyko’s Model

Why y = sine(latitude)?

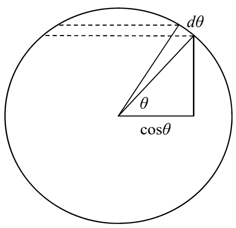
Archimedes



http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

Budyko’s Model

Why y = sine(latitude)?



surface area of a unit sphere

$$\int_{-\pi/2}^{\pi/2} 2\pi\cos\theta d\theta = 2\pi\sin\theta\Big|_{-\pi/2}^{\pi/2} = 4\pi$$

average over the sphere of a function of latitude $f(\theta)$

$$\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} f(\theta)2\pi\cos\theta d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} f(\theta)\cos\theta d\theta$$

(substitute $y = \sin(\theta)$) $= \frac{1}{2}\int_{-1}^{1} f(\arcsin y)dy$

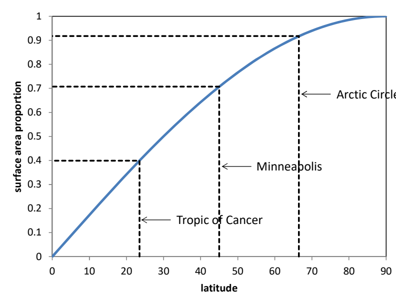
average over the sphere of a function $T(y)$

$$\overline{T} = \frac{1}{2}\int_{-1}^{1} T(y)dy$$

if T is symmetric across the equator: $\boxed{\overline{T} = \int_0^1 T(y)dy}$

Budyko’s Model

Why y = sine(latitude)?



Budyko’s Model

Budyko’s Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A + BT) + C(\overline{T} - T)$$

surface temperature; heat capacity; insolation; sin(latitude); albedo; OLR; heat transport; $\overline{T} = \int_0^1 T(y)dy$

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley’s quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$

Budyko’s Model

Next Week

The Dynamics of Budyko's Model

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A + BT) + C(\overline{T} - T)$$

surface temperature; heat capacity; insolation; sin(latitude); albedo; OLR; heat transport; $\overline{T} = \int_0^1 T(y)dy$