# Introduction to Multiflows

Richard McGehee
School of Mathematics
University of Minnesota

Mathematics of Climate Seminar
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MCRN
Mathematics and Climate Research Network

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## Multiflows

### Well-posed Problems

Jacques Hadamard

https://upload.wikimedia.org/wikipedia/commons/a/ae/Hadamard2_cropped.jpg

*According to Wikipedia*

The mathematical term ***well-posed problem*** stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:
**1.** a solution exists,
**2.** the solution is unique,
**3.** the solution's behaviour changes continuously with the initial conditions.

https://en.wikipedia.org/wiki/Well-posed_problem



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## Multiflows

### Well-posed Problems

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*Metaphysical Preference?*

https://en.wikipedia.org/wiki/Well-posed_problem



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## Multiflows

### Well-posed Problems

**Basic Theorem from Dynamical Systems**

If $f$ is a smooth vector field in Euclidean space, then the initial value problem

$$\frac{dy}{dt} = f(y), \quad y(0) = x$$

satisfies the following conditions:
**1.** a solution exists,
**2.** the solution is unique,
**3.** the solution's behavior changes continuously with the initial conditions.

*I.e.*, the initial value problem is **well-posed**.



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## Multiflows

### Well-posed Problems

**Basic Theorem from Dynamical Systems**

If $f$ is a *smooth* vector field in Euclidean space, then the initial value problem

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satisfies the following conditions:
**1.** a solution exists,
**2.** the solution is unique,
**3.** the solution's behavior changes continuously with the initial conditions.

What if $f$ isn't smooth? What if is it only continuous? What happens?



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## Multiflows

### Example from Sophomore Calculus

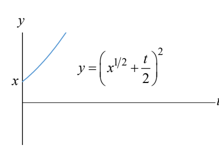
$$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$$

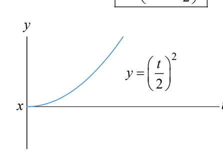
not smooth
not even Lipschitz continuous

Case: $y \ge 0$

$$\frac{dy}{dt} = y^{1/2}, \quad y^{-1/2}dy = dt, \quad \int y^{-1/2}dy = \int dt, \quad 2y^{1/2} = t + c$$

$$y(0) = x \;\Rightarrow\; 2x^{1/2} = c \;\Rightarrow\; 2y^{1/2} = t + 2x^{1/2} \;\Rightarrow\; \boxed{y = \left(x^{1/2} + \frac{t}{2}\right)^2}$$




Note that $y = 0$ is also a solution when $x = 0$.

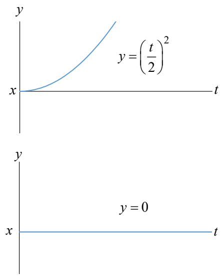
## Multiflows

$$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$$

There are two solutions satisfying $y(0) = 0$:

$$y(t) = \left(\frac{t}{2}\right)^2$$

$$y(t) = 0$$




Actually, there are an infinite number.

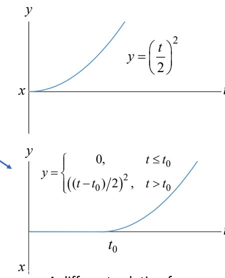


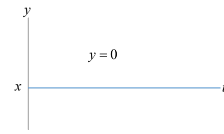
## Multiflows

$$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$$

This is a solution for any $t_0$:

$$y(t) = \begin{cases} 0, & t \le t_0 \\ ((t - t_0)/2)^2, & t > t_0 \end{cases}$$




A different solution for every $t_0$.



## Multiflows

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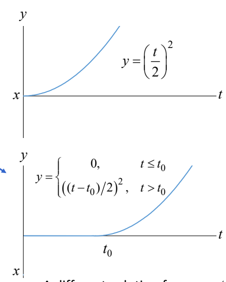
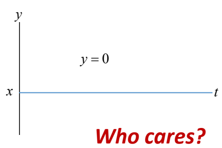



***Who cares?***

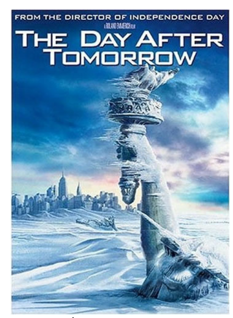
A different solution for every $t_0$.



## Multiflows

***Who cares?***

***Hollywood!***




**20th Century Fox 2004**



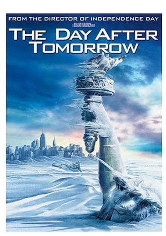
## Multiflows

**Dansgaard-Oeschger Events**

"Global warming" can cause the Northern Hemisphere to cool.

Melting glaciers can lower the salinity of the North Atlantic, causing a decrease in the flow of the Atlantic Meridional Overturning Circulation (AMOC), slowing the heat transfer to the Northern Hemisphere.

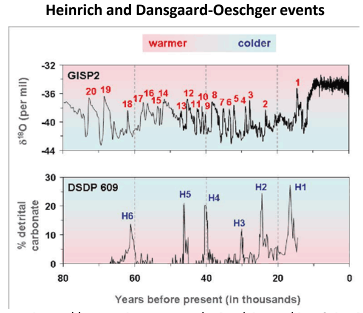
***This phenomenon happened twenty times during the last 80,000 years.***






## Multiflows

**Heinrich and Dansgaard-Oeschger events**




http://www.ncdc.noaa.gov/paleo/abrupt/data3.html

## Multiflows

**Current Atlantic Meridional Overturning Circulation weakest in last millennium**

The **Atlantic Meridional Overturning Circulation (AMOC)**—one of Earth's major ocean circulation systems—redistributes heat on our planet and has a major impact on climate. Here, we compare a variety of published proxy records to reconstruct the evolution of the AMOC since about AD 400. A fairly consistent picture of the AMOC emerges: **after a long and relatively stable period, there was an initial weakening starting in the nineteenth century, followed by a second, more rapid, decline in the mid-twentieth century, leading to the weakest state of the AMOC occurring in recent decades.**

NATURE GEOSCIENCE | VOL 14 | MARCH 2021 | 118–120 | www.nature.com/naturegeoscience118

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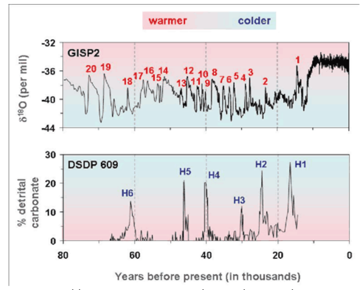
## Multiflows

**Heinrich and Dansgaard-Oeschger events**




http://www.ncdc.noaa.gov/paleo/abrupt/data3.html

***What caused the oscillations?***

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## Multiflows

***What caused the D-O oscillations?***

They could be self-oscillations in the natural dynamics of ocean circulation.

Pierre Welander, A simple heat-salt osciallator, *Dynamics of Atmospheres and Oceans* **6** (1982) 233-242.

R/V WeeLander is a 23-foot-long Beach Master work boat, informally named in honor of Professor Pierre Welander (1925—1996).

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## Multiflows

**Welander Model**

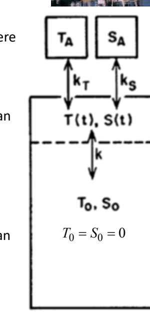
mixing with atmosphere

temperature → $\dot{T} = k_T(T_A - T) - k(\rho)T$

salinity → $\dot{S} = k_S(S_A - S) - k(\rho)S$

density → $\rho = -\alpha T + \gamma S$

mixing with deep ocean




Pierre Welander, *Dynamics of Atmospheres and Oceans* **6** (1982).

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## Multiflows

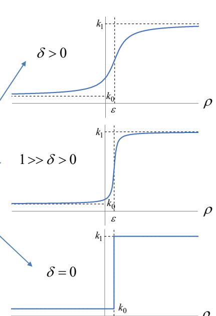
**Welander Model**

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

**The function *k***

$$k(\rho) = \frac{k_1 + k_0}{2} + \frac{k_1 - k_0}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{\delta}\right)$$

Limit as $\delta \to 0$ $\quad k(\rho) = \begin{cases} k_0, & \rho < \varepsilon \\ k_1, & \rho > \varepsilon \end{cases}$

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## Multiflows

**Welander Model**

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

Welander chose scientifically reasonable values and dimensionless variables and constants.

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1 - S) - k(\rho)S$$
$$\rho = -\alpha T + S$$

$$k(\rho) = \frac{1}{2} + \frac{2}{\pi}\tan^{-1}\left(\frac{\rho - \varepsilon}{\delta}\right)$$

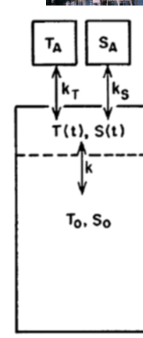
$\alpha = 0.8$

$\beta = 0.5$

$\varepsilon$ : small parameter

Limit as $\delta \to 0$ $\quad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$

Welander simulated the case $\delta = 0$ , finding a periodic orbit, and concluded the same would hold for $\delta > 0$.

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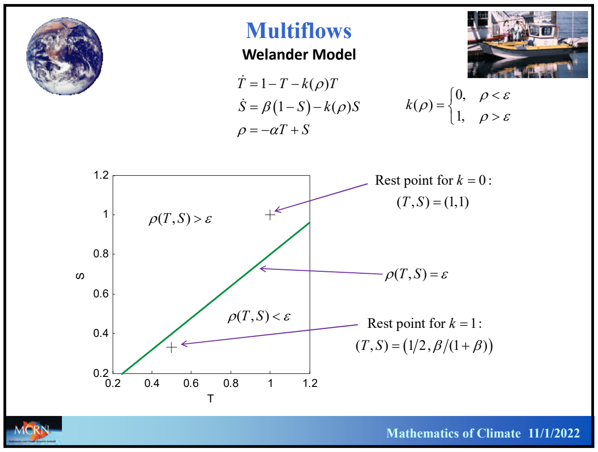
Multiflows
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
$\dot{S} = \beta(1 - S) - k(\rho)S$
$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\rho(T,S) > \varepsilon$
Rest point for $k = 0$: $(T,S) = (1,1)$
$\rho(T,S) = \varepsilon$
$\rho(T,S) < \varepsilon$
Rest point for $k = 1$: $(T,S) = (1/2, \beta/(1+\beta))$
T
S
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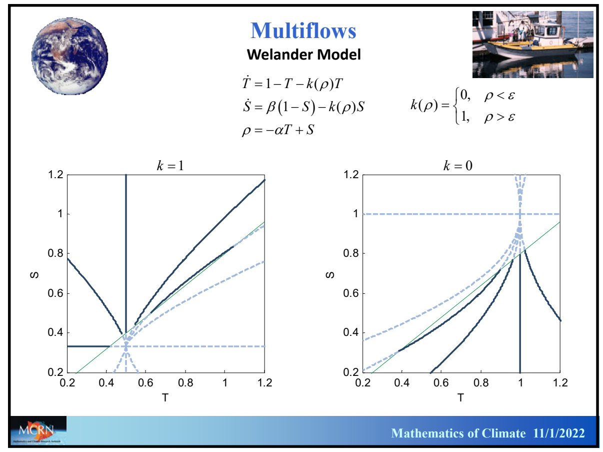
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$k = 1$
$k = 0$
T
S
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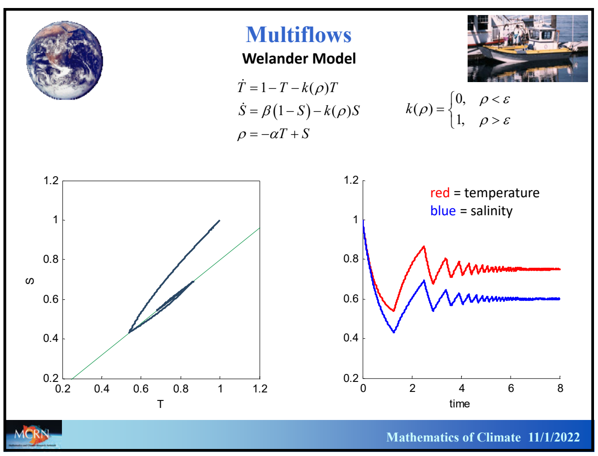
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red = temperature
blue = salinity
T
S
time
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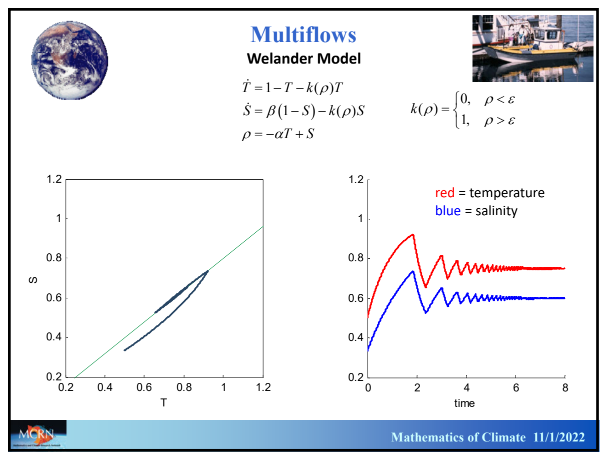
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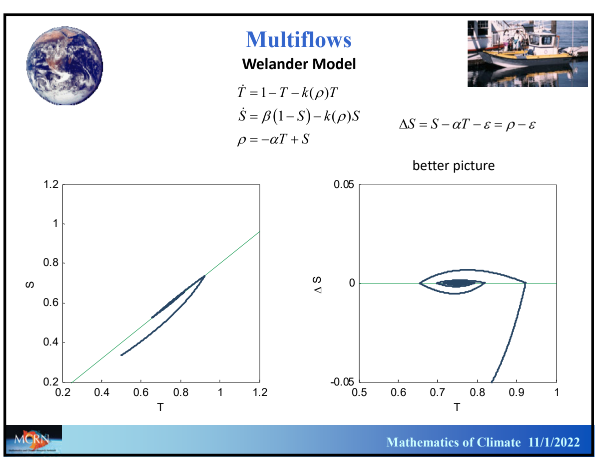
T
S
time
Mathematics of Climate 11/1/2022


Multiflows
Welander Model
$\dot{T} = 1 - T - k(\rho)T$
$\dot{S} = \beta(1 - S) - k(\rho)S$
$\rho = -\alpha T + S$
$\Delta S = S - \alpha T - \varepsilon = \rho - \varepsilon$
better picture
T
S
Δ S
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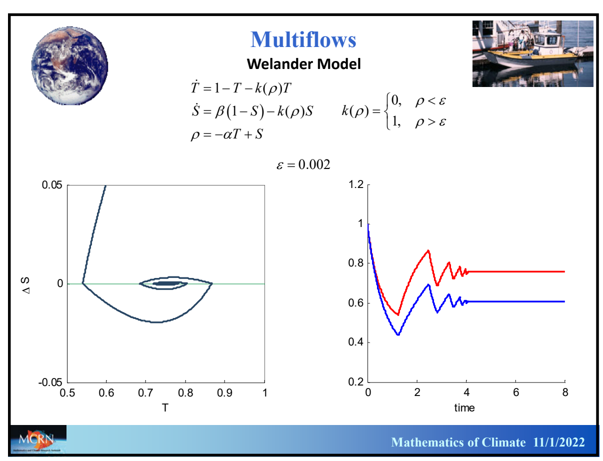
Multiflows
Welander Model
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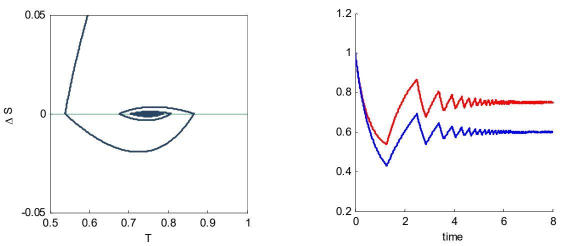
$\rho = -\alpha T + S$
$k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$
$\varepsilon = 0.002$
T
Δ S
time
Mathematics of Climate 11/1/2022

## Multiflows

### Welander Model

$$\dot{T} = 1 - T - k(\rho)T$$
$$\dot{S} = \beta(1-S) - k(\rho)S \qquad k(\rho) = \begin{cases} 0, & \rho < \varepsilon \\ 1, & \rho > \varepsilon \end{cases}$$
$$\rho = -\alpha T + S$$

$\varepsilon = 0$

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## Multiflows

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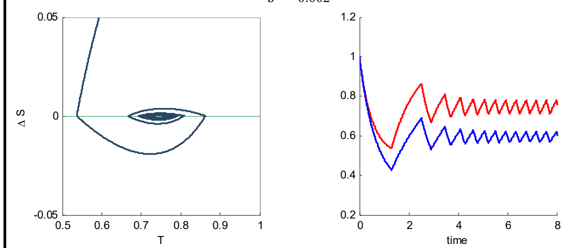
$$\rho = -\alpha T + S$$

$\varepsilon = -0.002$

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## Multiflows

### Welander Model

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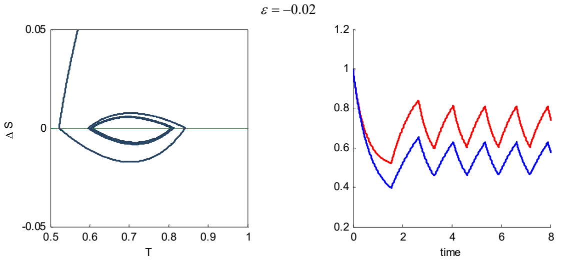
$$\rho = -\alpha T + S$$

$\varepsilon = -0.02$

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## Multiflows

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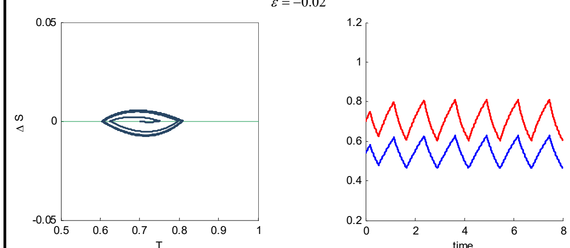


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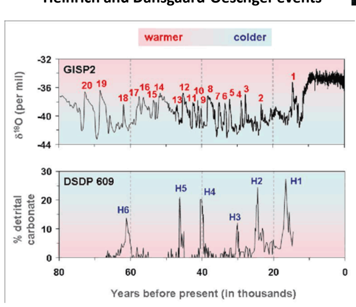
## Multiflows

### Heinrich and Dansgaard-Oeschger events




***Were these self-oscillations a la Welander?***

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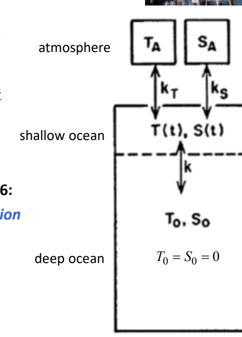
## Multiflows

Welander found the discontinuous limit easier to analyze. He assumed that the self-oscillations he found in the discontinuous limit held for the smooth system with a small parameter.

**Juliann Leifeld, 2016:**

***Welander's assumption was correct.***

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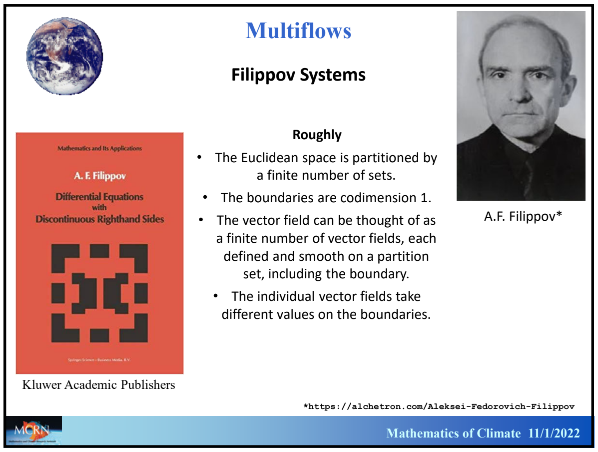
Multiflows
Filippov Systems
Mathematics and Its Applications
A. F. Filippov
Differential Equations with Discontinuous Righthand Sides
Kluwer Academic Publishers
Roughly
The Euclidean space is partitioned by a finite number of sets.
The boundaries are codimension 1.
The vector field can be thought of as a finite number of vector fields, each defined and smooth on a partition set, including the boundary.
The individual vector fields take different values on the boundaries.
A.F. Filippov*
*https://alchetron.com/Aleksei-Fedorovich-Filippov
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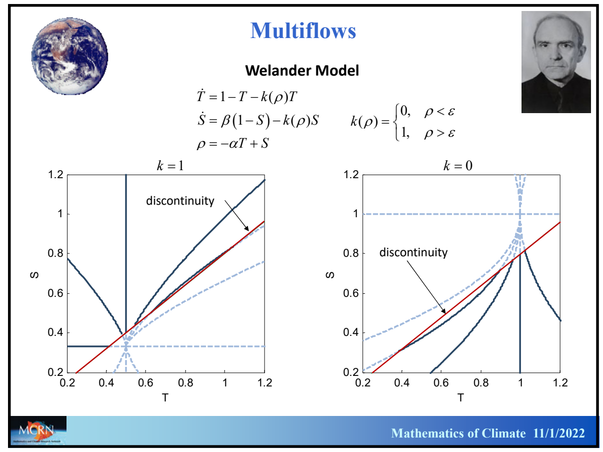
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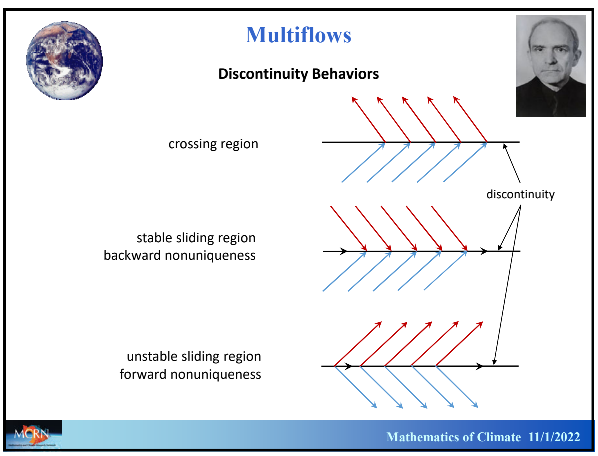
$k = 1$
$k = 0$
discontinuity
S
T
Mathematics of Climate 11/1/2022


Multiflows
Discontinuity Behaviors
crossing region
discontinuity
stable sliding region
backward nonuniqueness
unstable sliding region
forward nonuniqueness
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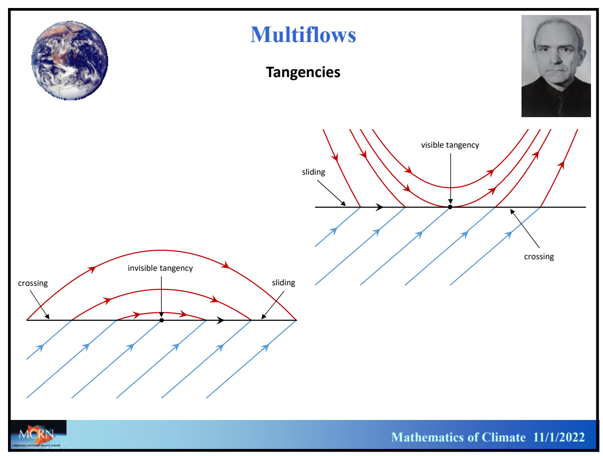
Multiflows
Tangencies
visible tangency
sliding
crossing
invisible tangency
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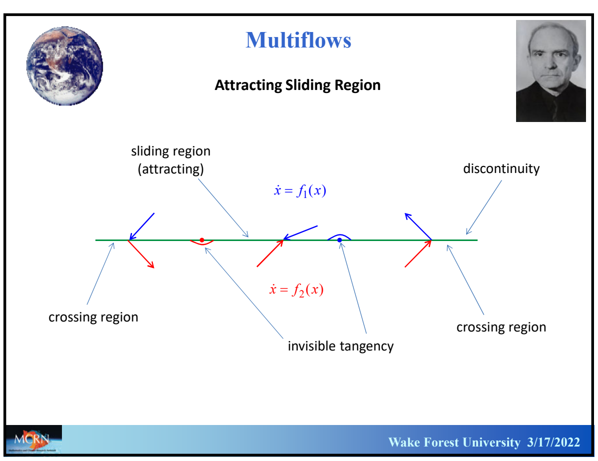
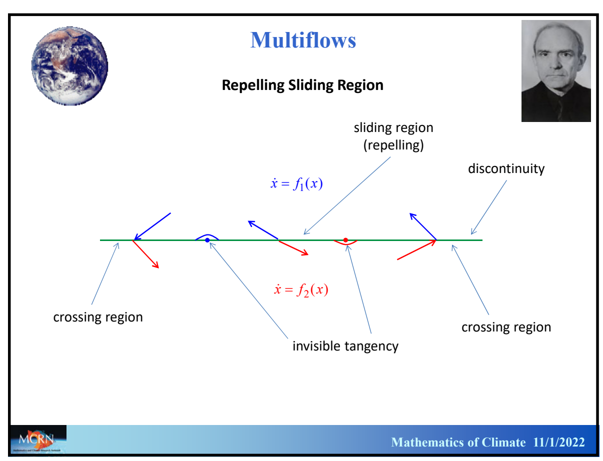

Multiflows
Attracting Sliding Region
sliding region
(attracting)
discontinuity
$\dot{x} = f_1(x)$
$\dot{x} = f_2(x)$
crossing region
invisible tangency
crossing region
Wake Forest University 3/17/2022


Multiflows
Repelling Sliding Region
sliding region
(repelling)
discontinuity
$\dot{x} = f_1(x)$
$\dot{x} = f_2(x)$
crossing region
invisible tangency
crossing region
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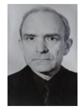
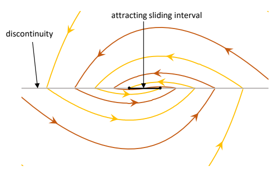
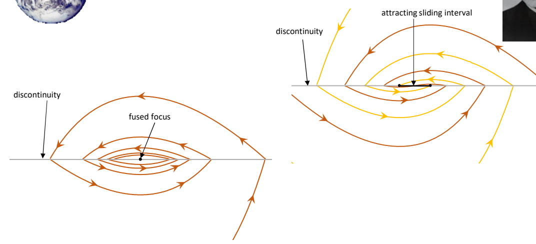
## Multiflows





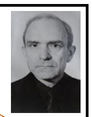


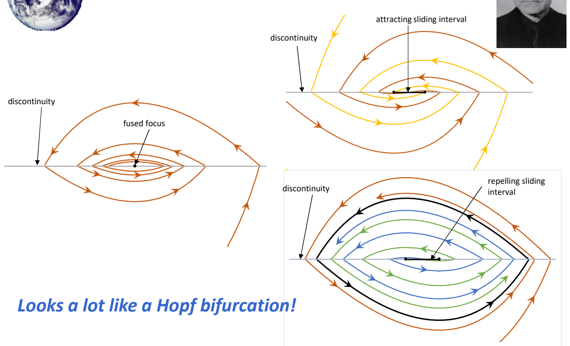

## Multiflows






## Multiflows






***Looks a lot like a Hopf bifurcation!***



## Multiflows

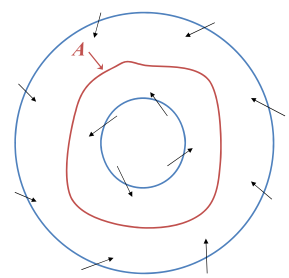
**Conley Index Theory**

Inside the annulus, there is an attracting set that "looks like" a circle.

**Question**

Is there a Conley theory for Filippov systems?

***If so, what good would it be?***






## Multiflows

### Relations

**Definitions**

A *relation* on a set $X$ is a subset of $X \times X$.

Let $f$ and $g$ be relations on a set $X$. The *composition* of $f$ and $g$ is the relation $f \circ g = \{(x,z) \in X \times X :$ there exists $y \in X$ with $(x,y) \in g$ and $(y,z) \in f\}$.

**Theorem**

If $f$ and $g$ are closed relations on a compact metric space $X$, then $f \circ g$ is closed.

**Note**

The space of closed relations on a compact metric space X, together with composition as the binary operator, is a semigroup with identity $\text{id} = \{(x,y) \in X \times X : x = y\}$.



## Multiflows

**Definition**

Let $X$ be a compact metric space.

A *multiflow* on $X$ is a subset $\Phi$ of $[0,\infty) \times X \times X$ satisifying

(1) $\Phi$ is closed,

(2) $\Phi^0 = \text{id}$,

(3) $\Phi^{t+s} = \Phi^t \circ \Phi^s$.

where

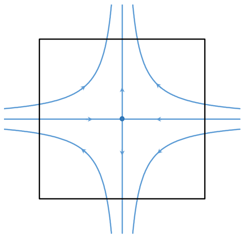
$$\Phi^t = \{(x,y) \in X \times X : (t,x,y) \in \Phi\}$$

**Note**

A multiflow $\Phi$ determines a semigroup homomorphism from $[0,\infty)$ to the space of closed relations: $t \mapsto \Phi^t$.

## Multiflow Examples

**Theorem:** (*Cameron Thieme 2020*) A flow restricted to a compact set is a multiflow.



McGehee & Sander, A new proof of the stable manifold theorem. *Z. angew. Math. Phys.* **47** (1996), 497-513.



## Multiflows

## Multiflow Examples

**Theorem:** (*McGehee 2018*) A continuous vector field generates a multiflow.

Lischitz continuous vector fields generate flows.
Continuous vector fields might not.
Why?
Uniqueness of solutions disappears.
Example:

$$\frac{dy}{dt} = |y|^{1/2}, \quad y(0) = x$$



## Multiflows

## Multiflow Examples

**Theorem:** (*Cameron Thieme 2020*) Filippov Systems generate multiflows.

**Theorem:** (*Kate Meyer 2019*) Certain control systems generate multiflows. Specifically, the system

$$\frac{dx}{dt} = f(x) + g(t), \quad x \in \mathbb{R}^n, f \text{ is smooth}, \|g\|_\infty \le r,$$

generates a multiflow.



## Multiflows

***So what?***

*I.e.*, so we show that something generates a multiflow. ***What good is that?***



## Multiflows
## Nonuniqueness

***So what?***

*I.e.*, so we show that something generates a multiflow. ***What good is that?***

***What is "good"?***

**Modeling Dogma**
A ***good*** model is well-posed.



## Multiflows

***So what?***

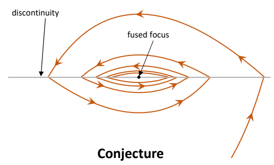
*I.e.*, so we show that something generates a multiflow. ***What good is that?***

***What is "good"?***

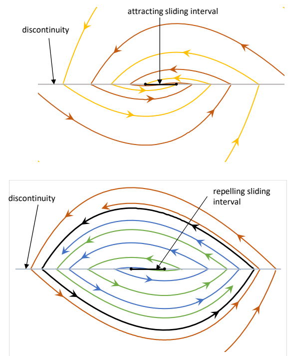
**Modeling Dogma**
A ***good*** model is well-posed.

**Heresy Avoidance**
A bad model is ***pretty good*** if it says something nice about a "nearby" good model.

## Multiflows




**Conjecture**
If a Filippov system is the continuous limit of a smooth system (in the "right" toplogy on multiflows), then the smooth system will have a nearby attractor limiting to the periodic orbit.
Sort of a Conley Hopf bifurcation.

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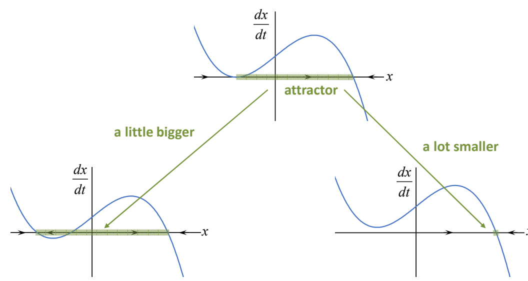
## Multiflows
### Semicontinuity of Attractors

As parameters change, attractors can get a little bigger or a little smaller or a lot smaller, but they can't get a lot bigger.

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## Nonuniqueness
**Multiflows**

**Cameron Thieme, PhD Thesis, 2020**

Thieme gave a definition of semicontinuity in a one-parameter family of multiflows.

Thieme gave conditions concluding that Conley Theory is valid for these one-parameter families.

Welander's model satisfies the Thieme's definition and conditions.

Welander's "nearby" smooth model has an attractor with the topology of (homotopic to) a circle.

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## Multiflows
**Open Questions**

Can Thieme's definition and conditions be generalized further?

Are there other interesting examples?
Artic sea ice?
Controlling epidemics?
Adaptive chemotherapy?

**McGehee's Pipedream**
Can singular perturbation problems (aka fast-slow systems) be formulated as multiflow systems?

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## Multiflows
**Well-posed Problems**

***According to Wikipedia***

The mathematical term ***well-posed problem*** stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:
**1.** a solution exists,
**2.** the solution is unique,
**3.** the solution's behaviour changes continuously with the initial conditions.

https://en.wikipedia.org/wiki/Well-posed_problem

**Jacques Hadamard**

https://upload.wikimedia.org/wikipedia/commons/a/ae/Hadamard2_cropped.jpg

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## Multiflows

**Not-Too-Badly Posed Problem**

The mathematical term ***well-posed problem*** stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:
**1.** a solution exists,
**2.** the solution is unique,
**3.** the solution's behaviour changes continuously with the initial conditions.

**Avoiding Heresy:**
A mathematical model is ***not-too-badly-posed*** if it can be described as a multiflow. I.e., uniqueness can be abandoned and continuity with initial conditions can be replaced by semicontinuity with nearby systems.

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Thank-you!
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