Permafrost Melt and the Heat Equation

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Permafrost

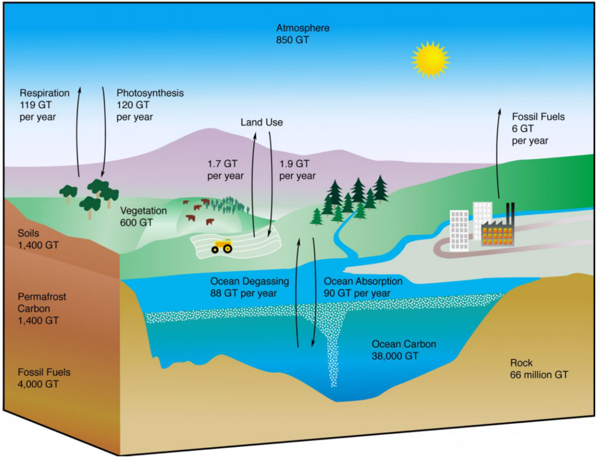
- The permanently frozen soils of the Arctic, known as permafrost, store large amounts of organic carbon, which accumulated over millennia due to slow decomposition in the cold Arctic region.
- Soil taxonomists define permafrost as material that remains at or below 0°C for two or more consecutive years.
- Permafrost thickness can range from one meter to more than 1,000 meters.
- Permafrost is composed of soil, organics, rock, and sand often with large blocks of iced mixed in.

Permafrost

Source: Natural Resources Defense Council

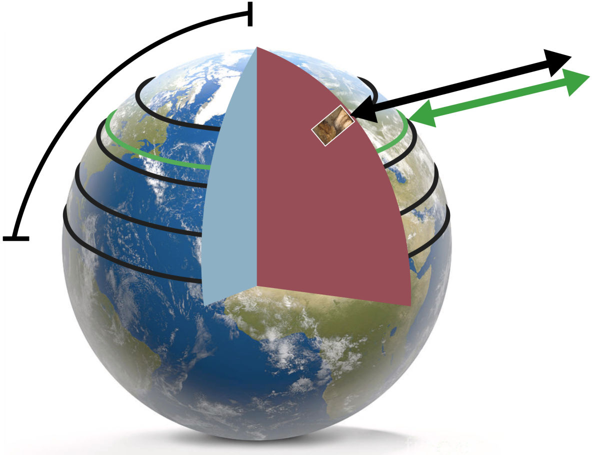
Permafrost: Source: Biskaborn, et al. 2019

Permafrost and Global Carbon Cycle



Source: National Snow and Ice Data Center.

Modeling Permafrost thawing and effects on the Global Carbon Cycle



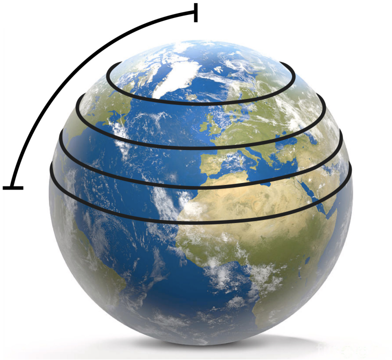
Overview

1. Budyko's Earth Energy Balance Model
2. Potential Carbon Emissions
3. Explicit model for permafrost melt
4. Future Work

Budyko's Earth Energy Balance Model

The differential-integral equation

$$R\frac{\partial T(y,t)}{\partial t} = (1-\alpha(y,\eta))Qs(y) - (A + BT(y,t)) + C(\bar{T}(t) - T(y,t))$$



Budyko's Earth Energy Balance Model

$$R\frac{\partial T(y,t)}{\partial t} = (1-\alpha(y,\eta))Qs(y) - (A + BT(y,t)) + C(\bar{T}(t) - T(y,t))$$

heat capacity **albedo** **insolation** **OLR** **heat transport**

Latitudinal distribution function (1), Albedo (2) and Global Mean Temperature (3)

$$s(y) \approx 1 - s_2(\frac{1}{2}(3y^2 - 1)) \tag{1}$$

$$\alpha(y) = \begin{cases} \alpha_1 & y < \eta \quad \text{[ice]} \\ \alpha_2 & y > \eta \quad \text{[no ice]} \end{cases} \tag{2}$$

$$\bar{T} = \int_0^1 T(y)dy \tag{3}$$

Variables and Parameters

The differential-integral equation

$$R\frac{\partial T(y,t)}{\partial t} = (1-\alpha(y,\eta))Qs(y) - (A + BT(y,t)) + C(\bar{T}(t) - T(y,t))$$

Variable	Value	Units	Description
t	-	year	Time
y	-	-	Sine of Latitude
T(t,y)	-	C°	Surface Temperature
Parameter	**Value**	**Units**	**Description**
R	-	$\frac{Wseconds}{m^2 C^\circ}$	Planetary Heat Capacity
Q	343	$\frac{W}{m^2}$	Insolation
s_2	0.482	dimensionless	Estimate on the effect due to obliquity on insolation
A	202	$\frac{W}{m^2}$	Temperature-independent outgoing longwave radiation
B	1.9	$\frac{W}{m^2 C^\circ}$	Temperature-dependent outgoing longwave radiation
C	3.04	$\frac{W}{m^2 C^\circ}$	Heat transport coefficient
α_1	.32	dimensionless	Albedo for latitudes south of snow line
α_2	.62	dimensionless	Albedo for latitudes north of snow line
T_c	-10	C°	Critical temperature at the snow boundary
η	sin(72)	sine of $72^\circ N$	Sine of Latitude of snow line

Equilibrium Temperature Profile

Budyko's Earth Energy Balance Model

$$R\frac{\partial T(y,t)}{\partial t} = (1-\alpha(y,\eta))Qs(y) - (A + BT(y,t)) + C(\bar{T}(t) - T(y,t))$$

Equilibrium Temperature solution

$$T(y) = \frac{1}{B+C}(Qs(y)(1-\alpha(y,\eta)) - A + C\bar{T})$$

Latitude of permafrost line

$$61^\circ N$$

Latitude of snow line

$$72^\circ N$$

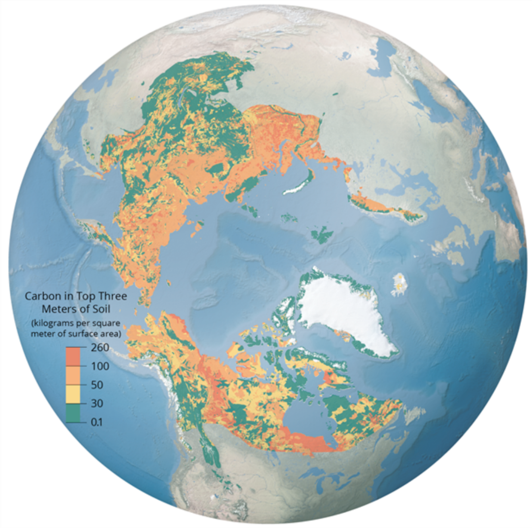
Potential Carbon Emissions

Environ. Res. Lett. **9** (2014) 085003 K Schaefer *et al*

Table 2. Projections of cumulative emissions from thawing permafrost, with CO_2 equivalents in parentheses[a].

Study	2100	Permafrost carbon emissions (Gt C)2200	2300	Flux uncertainty (%)	Temperature increase (K)2100	Initial carbon stock (Gt C)	Permafrost area loss (%)2100	Scenario
Zhuang *et al* (2006)[b]	37 (46)	na[c]	na	3%	na	na		A2
Dutta *et al* (2006)	40 (50)	na	na	na	na	460		5 °C Siberia
Burke *et al* (2013)	50 (62)[e]	na	99 (124)[e]	41%	na	850	76 ± 20	RCP8.5
Koven *et al* (2011)	62 (78)	na	na	11%	na	504	30	A2
Schneider von Deimling *et al* (2012)	63 (79)	302 (378)	380 (476)	16%	0.13 ± 0.10	800	57 ± 20	RCP8.5
Schuur *et al* (2009)[b]	85 (107)	na	na	15%	na	818		A2
Schaphoff *et al* [2013]	98 (122)	na	226 (283)[b]	23%	na	952	24	5 °C global
Gruber *et al* (2004)	100 (125)	na	na	na	na	400		2 °C global
Schaefer *et al* (2011)	104 (130)	190 (238)	na	36%	na	313	30 ± 10	A1B
Burke *et al* (2012)	150 (188)	na	na	67%	0.22 ± 0.14	951	65	RCP8.5
Schuur *et al* (2013)	158 (198)	na	345 (432)	24%	na	1488	55 ± 5[a]	RCP8.5
MacDougall *et al* (2012)	174 (218)	na	na	61%	0.27 ± 0.16	1026	56 ± 3	RCP8.5
Harden *et al* (2012)	218 (273)[e]	na	436 (546)[e]	85%	na	1060	74	RCP8.5
Raupach and Canadell (2008)[d]	347 (435)	na	na	na	0.7	500		A2

Carbon in Top Three Meters of Soil: Schuur 2019



Potential Carbon Emissions

$$(\text{Total Carbon in Permafrost})\left(\frac{\text{Change in Permafrost Surface Area}}{\text{Original Permafrost Surface Area}}\right) = \text{Carbon emissions}$$

Receding Permafrost line

Potential Carbon Emissions

- To estimate how much carbon would be released from the permafrost if the global mean temperature rose by 2 recall that the surface area is proportional to y, the sine of the latitude. With a current permafrost line at y_p we have that the proportion of the globe covered by permafrost is $1 - y_p = 0.125$
- Then the proportion of the permafrost melted is given by:

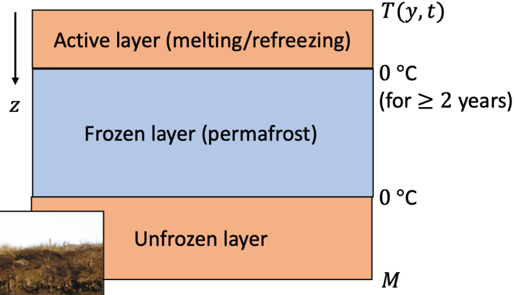
$$\frac{\Delta y}{1 - y_p} = \frac{0.027}{0.125} = 0.216$$

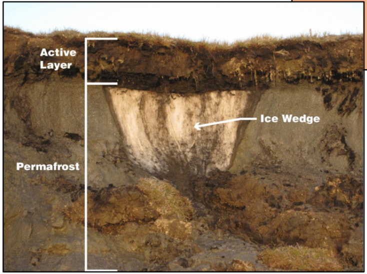
- Therefore an estimate of amount of carbon released

$$1400\frac{\Delta y}{1 - y_p} = 1400\frac{0.027}{0.125} = \mathbf{302}\text{GtC}$$

- In comparison, the total fossil fuel emission since 1751 are 375 GtC.

Explicit model for permafrost melt

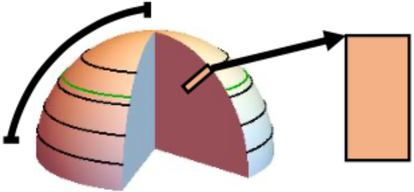




Explicit model for permafrost melt

Heat Equation

$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t > 0, \quad 0 \leq z \leq l$$



Explicit model for permafrost melt

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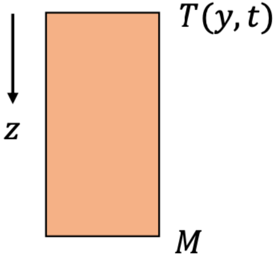


Explicit model for permafrost melt

Heat Equation Version 2

$$\frac{\partial E}{\partial t} = \frac{\partial q}{\partial z}$$

$$q = \kappa(T)\frac{\partial T}{\partial z}$$

$$\kappa(T) = \begin{cases} \kappa_1 & T \geq 0 \\ \kappa_2 & T < 0 \end{cases}$$

Explicit model for permafrost melt

Heat Equation

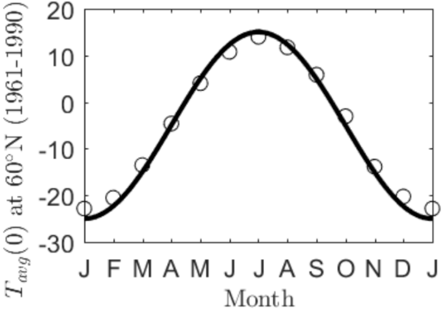
$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t > 0, \quad 0 \leq z \leq l$$

Variable/ Parameter	**Value**	**Units**	**Description**
z	-	m	Soil depth
l	1,000	m	Depth assumption
M	[10,35]	C°	Heat source range from the convective portion of the mantle
M	60	C°	Heat source from the convective portion of the mantle used in simulation
k	[75,828]	C°	Thermal diffusivity range
$k = \frac{K}{\rho c}$	700	C°	Thermal diffusivity used in simulation
ρc	0.5	$\frac{cal}{cm^3 K}$	Volumetric heat capacity of the medium
K	[5,55]	$\frac{W}{mK}$	Thermal conductivity
F	0	C°	Temperature forcing

Explicit model for permafrost melt

Surface Boundary Condition via sinusoidal fit

$$T_y(0, t; y) = -5 - 20\cos(2\pi t) + F$$



Source: CRU CL v2.0

Explicit model for permafrost melt

Heat Equation

$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t > 0, \quad 0 \leq z \leq l$$

Surface Boundary Condition via sinusoidal fit

$$T_y(0,t;y) = -5 - 20\cos(2\pi t) + F$$

Lower Boundary Condition

$$T_y(l,t;y) = \mathrm{M} = 60^\circ\mathrm{C}$$

Explicit model for permafrost melt

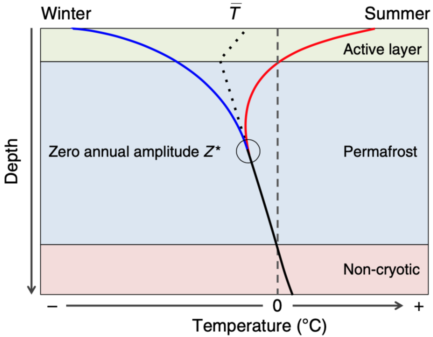
Heat Equation

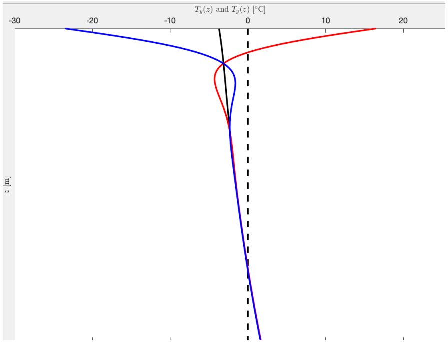
$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t > 0, \quad 0 \leq z \leq l$$

Linear Initial Condition

$$T_y(z,0) = \frac{M - T(y,0)}{l}z + T(y,0)$$

Temperature Profile Permafrost

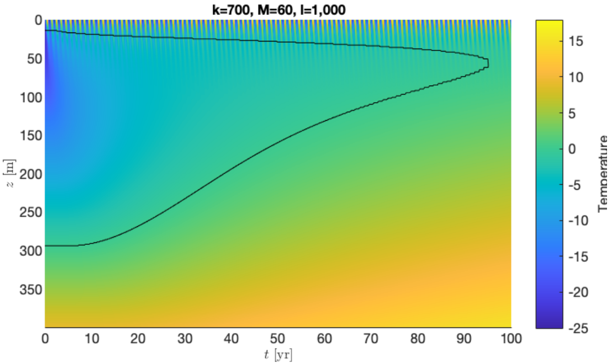




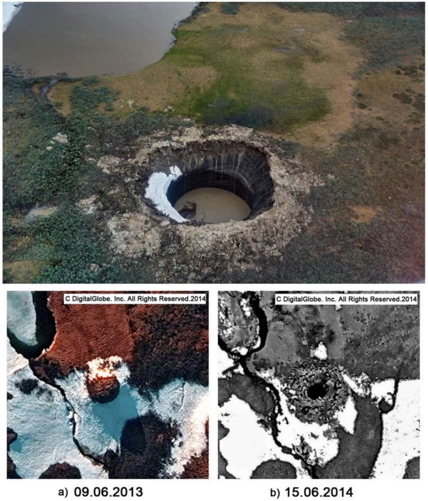
Explicit model for permafrost melt

Heat Equation

$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t>0, \quad 0 \leq z \leq l$$



Permafrost Crater



Source: Buldovicz 2018

Permafrost Crater

"Crater 1" - the first reported crater in 2014 on the Yamal peninsula. Source: Forbes

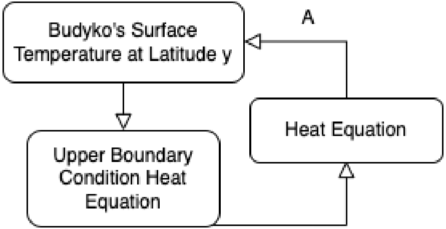
Future Work

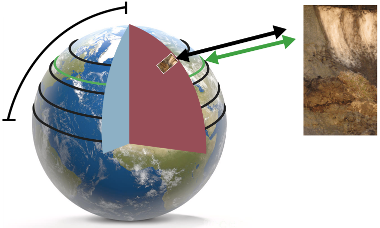
Budyko with Heat Equation

$$R\frac{\partial T(y,0,t)}{\partial t} = (1-\alpha(y,\eta))Qs(y) - (A + BT(y,0,t) + C(\bar{T}(z,t) - T(y,0,t))$$

$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t > 0, \quad 0 \leq z \leq l$$

Future Work


Budyko's Surface Temperature at Latitude y
A
Heat Equation
Upper Boundary Condition Heat Equation



Coefficients for Outgoing Longwave Radiation

- Caldeira and Kasting investigated the effect of varying amounts of carbon dioxide concentration in the atmosphere, measured by its partial pressure, pCO_2, on the outgoing longwave radiation terms. They used a climate model that simulates the vertical profile of atmospheric temperature under the assumption of radiative–convective equilibrium.
- Using results from 2,000 calculation rounds of this radiative-convective model with different carbon dioxide partial pressures, they fitted the constants A and B as a function of $\varphi = \ln(\frac{pCO_2}{(pCO_2)_{ref}})$, where $(pCO_2)_{ref}$ is a reference value corresponding to the present value of CO_2 at 300 parts per million.

Coefficients for Outgoing Longwave Radiation

Polynomial fit

$$A = -326.4 + 9.161\varphi - 3.164\varphi^2 + 0.5468\varphi^3$$

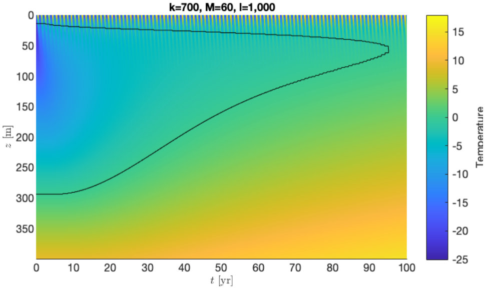
$$B = 1.953 - 0.04866\varphi + 0.01309\varphi^2 - 0.002577\varphi^3$$

Current Work

Temperature independent outgoing longwave radiation

$$\bar{A} = A - \int f(G(y))dy$$

$$G(y) = \begin{cases} -Dz_p(y) + \alpha & z_p(y) \geq 0 \\ 0 & z_p(y) < 0 \end{cases}$$



Current work

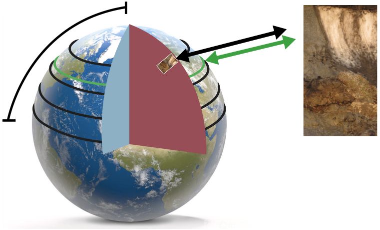
- With inspiration from the work of Caldeira and Kasting, we propose to use satellite data to compute the coefficients for outgoing longwave radiation.
- Exploring other formulations for the heat equation.

Current Work

Budyko with Heat Equation

$$R\frac{\partial T(y,0,t)}{\partial t} = (1-\alpha(y,\eta))Qs(y) - (A + BT(y,0,t) + C(\bar{T}(z,t) - T(y,0,t))$$

$$\frac{\partial T_y(z,t)}{\partial t} = k\frac{\partial^2 T_y(z,t)}{\partial z^2}, \quad t > 0, \quad 0 \leq z \leq l$$



References

Kaitlin Hill and Richard McGehee (2018)
Heat conduction through permafrost and its potential for explosive behavior
preprint arXiv:1810.12370

John Nguyen and Aileen Zebrowski (2020)
Adapting the Budyko Model to Analyze Permafrost Recession and Potential for Carbon Feedback
SIAM Undergraduate Research Online volume 14

Boris K Biskaborn, et al (2019)
Permafrost is warming at a global scale
Nature communications, volume 10

Richard McGehee and Esther Widiasih (2014)
PA quadratic approximation to Budyko's ice-albedo feedback model with ice line dynamics
SIAM Journal on Applied Dynamical Systems, volume 13