A Filippov Framework for an Extended Budyko Model

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Abstract

In latitude-dependent energy balance models, ice-free and ice-covered conditions form physical boundaries of the system. With carbon dioxide treated as a bifurcation parameter, the resulting bifurcation diagram is nonsmooth with curves of equilibria and boundaries forming corners at points of intersection. Over long time scales, atmospheric carbon dioxide varies dynamically and the nonsmooth diagram becomes a set of quasi-equilibria. In this article, we extend an energy balance model to include slowly varying carbon dynamics in a way that allows for a nonsmooth dynamical systems treatment. In our analysis within this framework, we prove existence and forward uniqueness of solutions and show that the physical region is forward invariant with sliding motion on the physical boundaries.

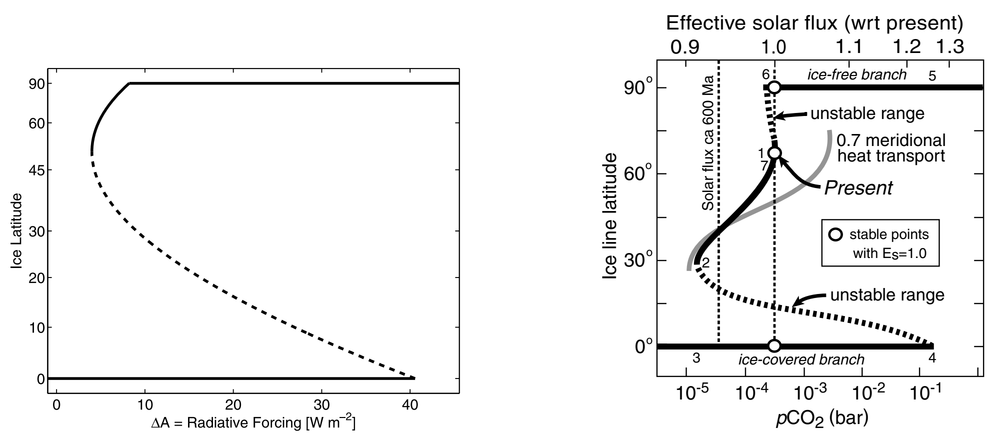
1 Introduction

There is significant evidence that large glaciation events took place during the Proterozoic era (2500-540 million years ago). In particular, this evidence points to the existence of glacial formations at low latitudes, see the review articles [15, 23] and the references therein. One theory on the exodus from such an extreme climate was put forward by Joseph Kirschvink [18], who advocated that there was accumulation of greenhouse gases in the atmosphere, e.g. CO_2. His theory purports that during a large glaciation chemical weathering processes would be shut down, thus eliminating a CO_2 sink. Moreover, volcanic activity would continue during the glaciated state. The combination of these effects would *slowly* lead to enough build-up of atmospheric carbon dioxide to warm the planet and start the melting of the glaciers. Once a melt began, a deglaciation would follow *rapidly* due to ice-albedo feedback.

There has been a wealth of modelling work on "snowball" events ranging from computationally intensive global circulation models (GCMs) to low dimensional conceptual climate models (CCMs). In 1969, Mikhail Budyko and William Sellers independently proposed energy balance models (EBMs); these were CCMs capturing the evolution of the temperature profile of an idealized Earth [3, 25]. Many others, for example [5, 7, 22], have followed in the footsteps of Budyko and Sellers and used similar conceptual models capable of exhibiting snowball events. The low dimensionality of CCMs allows for a dynamical systems analysis, and hence a deeper investigation into some of the key feedbacks such as greenhouse gas and the ice-albedo effect.

From the point of view of dynamical systems, many of these early works share a similar theme by focusing on a bifurcation analysis with respect to the *radiative forcing* parameter, one that depends on changes in atmospheric CO_2 and other greenhouse gas levels. The reader may find figures similar to those displayed in Figure 1 in earlier works [7, 15, 24]. These figures illustrate the glaciation state of an Earth with symmetric ice caps in terms of atmospheric CO_2 i.e. ice line latitude at 90^o means Earth is ice free while 0^o means Earth is fully glaciated. In both figures, the effect of the radiative forcing due to CO_2 and other greenhouse gases is treated statically as a parameter in a simple bifurcation analysis. Mathematically, the bifurcation analysis already extends beyond the realm of smooth dynamical systems. On one hand, the bifurcation diagrams are obtained conventionally, with the bold curves indicating the stable branches and the dotted curves showing the unstable ones. On the other hand, the extreme states (ice-free and ice-covered) are not true equilibria of the system, but treated as such in the literature as evidenced by the labeling of ”ice-free branch” and ”ice-covered branch”. Indeed, the extreme states of the ice line are special because they serve as physical boundaries for the dynamics: the ice line cannot extend beyond the pole nor, because of the north-south symmetry assumed in the models, can it move downward of the equator. Therefore, mathematical models of snowball earth must reflect this physical imposition and be treated with a nonsmooth systems perspective.

Since glacial extent varies over time and dynamic processes such as chemical weathering affect the level of atmospheric CO_2, the bifurcation diagrams in Figure 1 can be viewed as phase planes with dynamic variables consisting of the glacier extent (ice line) and radiative forcing due to greenhouse gases. In particular, if a global glaciation did occur, and Kirschvink’s argument about the accumulation of atmospheric CO_2 held, then we should expect orbits of this dynamical system to traverse the extreme ice latitudes, i.e. the equator and the pole. Following Kirschvink’s idea further, we treat greenhouse gas effects on the energy balance by incorporating a slow CO_2 variable and treat the ice latitude as a relatively fast variable. The goal of the present article is to analyze such an interplay using a nonsmooth systems framework, thus providing mathematical support for Kirschvink’s hypothesis. In this work, we treat the bifurcation diagram (similar to those in Figure 1) as a set of quasi-steady states of a slow-fast system and utilize the theory of Filippov.



(a) Figure 10 from Abbot, Voigt, and Koll [7] (b) Figure 6 from Hoffman and Schrag [15]

Figure 1: Bifurcation diagrams from energy balance models illustrating hysteresis in the climate system. In each figure, solid lines correspond to stable steady states while dashed lines correspond to unstable steady states. The positive horizontal axis can be thought of as increasing atmospheric carbon dioxide, and the vertical axis is the latitude of the ice line. Stability of snowball and ice-free states is *inferred*; these are physical boundaries and not true equilibria of the models.

We present the topics as follows. In the next section, we motivate the model of interest. In Section 3, we propose an extension of the model that is amenable to a nonsmooth dynamical systems treatment. In particular, we use Filippov’s theory for differential equations with discontinuous right-hand sides to show that an ice line model based on a latitude-dependent EBM coupled with a simple equation for greenhouse gas evolution can be embedded in the plane to form a system that has unique forward-time solutions. We end Section 3 with an analysis of the system dynamics and conclude the paper with a discussion in Section 4.

2 The Equations of Motion

2.1 The Budyko Energy Balance Model and the Ice Line Equation

The Budyko EBM describes the evolution of the annual temperature profile $T = T(y,t)$, where t denotes time and y denotes the sine of the latitude. The governing equation may be written:

$$R\frac{\partial}{\partial t}T(y,t) = Qs(y)(1-\alpha(\eta,y)) - (A + BT(y,t)) + C\left(\int_0^1 T(y,t)dy - T(y,t)\right). \tag{1}$$

The main idea is that the change in temperature is proportional to the imbalance in the energy received by the planet. The amount of short wave radiation entering the atmosphere is given by $Qs(y)(1-\alpha(\eta,y))$, where Q is the total solar radiation (treated as constant), $s(y)$ is the distribution of the solar radiation, and $\alpha(\eta,y)$ is the albedo at latitude y given that the ice line is at latitude η. The outgoing longwave radiation is the term $A + BT(y,t)$; it turns out that the highly complex nature of greenhouse gas effects on the Earth's atmosphere can be better approximated by a linear function of surface temperature than by the Stephan-Boltzmann law for blackbody radiation (σT^4), see the discussion in [14]. The parameter A is particularly interesting here, because it is related to greenhouse gas effects on the climate system, which we will describe further in Section 2.2. The transport term $C(\int_0^1 T(y,t)dy - T)$ redistributes heat by a relaxation process to the global average temperature. Assuming symmetry of the hemispheres, one may take $y \in [0,1]$ and $T(y,t)$ as a symmetric temperature profile over the interval $[-1,1]$, hence, an even function of y. A more detailed discussion of this model can be found in [28]. For interested readers, Table 1 lists the polynomial expressions of some key functions in this model using standard parameter values.

The evolution of the ice-water boundary, or what is often called the *ice line*, η, affects the albedo function $\alpha(\eta,y)$. In [28] and [22], a critical ice line annual average temperature, T_c, is specified. Above this temperature ice melts, causing the ice line to retreat toward the pole, and below it ice forms, allowing glaciers to advance toward the equator. One way to model this is described in [31], where the augmented ice line equation governing η is written as

$$\frac{d\eta}{dt} = \rho(T(\eta) - T_c). \tag{2}$$

The exact value of ρ is unspecified in our analysis, except that it is assumed to be large in comparison to the timescale governed by volcanism and weathering processes. The interested reader will find a detailed discussion of the value of ρ in [21]. Because (1) is an integro-differential equation, the phase space must contain the function space of the temperature profile. Therefore, the dynamics of this coupled system take place in some infinite-dimensional space. The work by Widiasih in [31] treated the model in a discrete time framework and showed the existence of a one-dimensional attracting manifold.

McGehee and Widiasih [21] imposed equatorial symmetry by using even Legendre polynomials with a discontinuity at the iceline $y = \eta$, and showed the existence of a one-dimensional attracting invariant manifold, similar to that shown in [31]. The invariant manifold is parametrized by the ice line, thereby reducing the dynamics to a single equation:

$$\frac{d\eta}{dt} = h_0(\eta; A) \tag{3}$$

with

$$h_0(\eta; A) := \rho\left(\frac{Q}{B+C}\left(s(\eta)(1-\alpha(\eta,\eta)) + \frac{C}{B}(1-\overline{\alpha}(\eta))\right) - \frac{A}{B} - T_c\right) \tag{4}$$

where $\overline{\alpha}(\eta) = \int_0^1 s(y)\alpha(\eta, y)dy$. The parameter A appearing in (3) is indeed the greenhouse gas parameter in [21], and is playing a similar role as that in the bifurcation diagram shown in Figure 1a.

In what follows, we apply the invariant manifold result of McGehee and Widiasih [21] by coupling the ice line equation (3) to an equation for atmospheric greenhouse gas evolution through the (former) parameter A. From here on, we highlight the explicit dependence of the equation for η on A by writing

$$\frac{d\eta}{dt} = h(A, \eta)$$

where $h(A, \eta) = h_0(\eta; A)$ and $h(A, \eta)$ is thought of as a real valued function over the plane $\mathbb{R} \times \mathbb{R}$.

Parameters	Value	Units	Functions
Q	321	Wm^{-2}	$s(y) = 1 - \frac{0.482}{2}(3y^2 - 1)$
s_1	1	dimensionless	
s_2	-0.482	dimensionless	$h(A, \eta) = \rho\left(112.88 + 56.91\eta - 24.31\eta^2 - 11.05\eta^3 - \frac{A}{1.5}\right)$
B	1.5	$\mathrm{Wm}^{-2}\mathrm{K}^{-1}$	
C	2.5B	$\mathrm{Wm}^{-2}\mathrm{K}^{-1}$	$g(A, \eta) = \delta(\eta - \eta_c)$
α_1	0.32	dimensionless	
α_2	0.62	dimensionless	$\alpha(\eta, y) = \begin{cases} \alpha_1 \text{ when } y < \eta \\ \frac{\alpha_1+\alpha_2}{2} \text{ when } y = \eta \\ \alpha_2 \text{ when } y > \eta \end{cases}$
T_c	-10	°C	

Table 1: Parameter values as in [7] and functions as in [21].

2.2 Incorporating Greenhouse Gases

We now make an argument for a very simple form of an equation for greenhouse gas evolution. In the Budyko and Sellers models [3, 25], the parameter A plays the important role of reradiation constant. The Earth absorbs shortwave radiation from the sun, and some of this is reradiated in the form of longwave radiation. The current value of A is measured using satellite data to be approximately $202\ W/m^2$ [14, 28].

However, throughout the span of millions of years A is not constant, as the amount of heat reradiated to space depends crucially on greenhouse gases, especially carbon dioxide. In fact, it was posited in [5] that A behaves like a function of the logarithm of atmospheric carbon dioxide, measured in parts per million. Intuitively, adding carbon dioxide to the atmosphere decreases its emissivity, allowing less energy to escape into space. Therefore, outgoing longwave radiation should vary inversely with CO_2.

Since the land masses were concentrated in middle and low latitudes prior to the global glaciation period, Kirschvink postulated that the stage was set for an ice-covered Earth. Then 'On a snowball Earth, volcanoes would continue to pump CO_2 into the atmosphere (and ocean), but the sinks for CO2 silicate weathering and photosynthesis would be largely eliminated '[15, 18]. In more recent work, Hogg [16] put forth an elementary model for the evolution of greenhouse gases consistent with Kirschvink's theory. In short, he argued that the main sources for atmospheric CO_2 were due to an averaged volcanism rate and ocean outgassing. The main carbon dioxide sink was said to be due to the weathering of silicate rocks. For our purposes, it will be enough to work with volcanism and weathering. Let V denote the rate of volcanism and W the weathering rate, where the latter is assumed to depend on the location of the iceline, or more specifically the amount of available land or rock to be weathered. Putting this together with the assumption that the reradiation variable A varies inversely with CO_2 gives

$$\begin{aligned}\frac{dA}{dt} &= -(V - W\eta) \\ &:= \delta(\eta - \eta_c). \end{aligned} \tag{5}$$

where $\delta > 0$ and $\eta_c = \frac{V}{W}$, the ratio of volcanism to weathering. Let $g(A, \eta) := \delta(\eta - \eta_c)$.

Remark 2.1. *In what follows, we often replace the right hand side of equation* (5) *with the more general function* $g(A, \eta)$. *While our analysis in Sections 3.3 and 3.4 uses the linear function of ice line alone, it is conceivable that greenhouse gas feedbacks depend directly on their atmospheric concentration, see eg. [5, 23]. We leave this avenue for future exploration and extension of the current work.*

Next, allowing A to vary in the ice line equation (4), we now have a system of equations for A and η:

$$\begin{cases} \dot{A} = g(A, \eta) \\ \dot{\eta} = h(A, \eta). \end{cases} \tag{6}$$

To understand the timescale of A, we refer to the elucidation of the snowball scenario by Hoffman and Schrag in which they argue that weathering took place at a much slower rate than the ice-albedo feedback (see Fig. 7 on page 137 [15]).

The idea of packaging the essence of the long term carbon cycle into one simple equation is not novel, and is consistent with earlier findings [10, 16, 20]. The novelty comes from connecting such an equation to an energy balance model and analyzing the dynamics of the coupled system. Indeed, the parameter A_{ex} on page 644 [20] is the variable η in equation (6) and it represents *the effective area of exposure of fresh minerals*. The parameter η_c can therefore be thought of as a *critical* area. The coupling of the ice line η and the greenhouse gas variable A follows naturally.

In its current form, the model does not restrict dynamics to the physical region $0 \leq \eta \leq 1$. There are orbits that exit this interval on either end, and we must therefore create reasonable assumptions for projection of this motion onto the boundaries. Furthermore, η should evolve along the physical boundaries, as suggested by the bifurcation diagrams in Figure 1. This will be guaranteed by first carefully extending the vector field to the whole plane, and then utilizing Filippov's theory for the resulting nonsmooth system.

3 A Filippov System for a Glaciated Planet

Before we dive into the analysis of the coupled (A, η) system, we shall first build an intuition for the type of system of our focus and prove a general result for such a system. We then apply this result to an extended version of (6) and study its dynamics. We find that this framework both ensures that trajectories with initial conditions in the physical region remain in this region for all time and produces the expected sliding motion on the boundaries.

Various frameworks for analyzing non-smooth systems have been developed, eg. by Caratheodory, Rosenthal, Viktorovskii, Utkin [12, 6]. Here, we choose to use the framework developed by A.F. Filippov [12].

3.1 Casting the Model in a Filippov framework

As mentioned previously, the pole and the equator define physical constraints of the ice line; it must stay in the unit interval $[0, 1]$. However, the model as it stands does not take this into account. A natural choice is to assume that $\dot{\eta} = 0$ when η reaches a physical boundary. In addition to this, one needs to account for the stability/instability of these states, thus allowing orbits to exit the boundary when it becomes unstable. More specifically, we should take

$$\begin{cases} \dot{\eta} &= \begin{cases} 0 & \{\eta = 0 \text{ and } h(\eta, A) < 0\} \text{ or } \{\eta = 1 \text{ and } h(A, \eta) > 0\} \\ h(A, \eta) & \text{otherwise} \end{cases} \\ \dot{A} &= g(A, \eta) \end{cases} \tag{7}$$

The first equation forces η to stop at the physical boundary exactly when it is about to cross that boundary. When this happens, η should remain constant while A continues to evolve. As A evolves, so does the value of h and this should result in the $\dot{\eta}$ equation "turning back on" as soon as h becomes positive on the lower boundary, $\eta = 0$, or negative on the upper boundary, $\eta = 1$. This is analogy with Kirschvink's hypothesis, as we expect orbits that enter the snowball state to recover when enough carbon dioxide has built up in the atmosphere. However, it is not clear that classical results from dynamical systems apply because the system is nonsmooth as well as undefined beyond the physical boundary. Fundamentally, the existence and uniqueness of solutions needs to be verified. Our analysis takes the approach of *extending* the vector field to the nonphysical region in a way that makes the system amenable to the theory of Filippov. This mathematical extension is used to obtain exactly the expected dynamics of the system (7) outlined above, including boundary motion. There are many possible modeling choices for the boundary, however, as will be discussed in Remark 3.3, our proposed extension ensures the consistency of the model dynamics with the existing snowball theory demonstrated by the literature e.g. in [7, 5]. The type of extension we adopt is highlighted in Section 3.2 below. Much of the proof relies on the theory developed by Fillipov [12, 13].

3.2 Existence and uniqueness of solutions

The following general result will motivate and apply to an extended version of the system (6), as well as guarantee the desired motion described in (7).

Theorem 3.1. *Let $G(x, y), H(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions. Define the vector field on $\mathbb{R} \times \mathbb{R}$ as follows:*

$$\dot{x} = G(x, y) \tag{8}$$

$$\dot{y} = \begin{cases} -|H| & \textit{when } y > 1 \\ \frac{H-|H|}{2} & \textit{when } y = 1 \\ H & \textit{when } 0 < y < 1 \\ \frac{H+|H|}{2} & \textit{when } y = 0 \\ |H| & \textit{when } y < 0. \end{cases} \tag{9}$$

Then, an initial value problem with time derivative (8)-(9) *has a unique forward-time Filippov solution. Furthermore, the strip $\mathbb{R} \times [0, 1]$ is forward invariant.*

Proof. First, to simplify notation, let $F(x,y)$ denote the vector-valued function $[G(x,y), H(x,y)]$. To show the existence of a Filippov solution, one must show that F is locally essentially bounded or equivalently, satisfies what Filippov called *Condition B* on p. 207 of [12]. This is straight forward to check since F is Lipschitz continuous everywhere except at the boundaries $y = 0, 1$.

To show uniqueness of the Filippov solution, we must show the *one-sided Lipschitz condition*, ie. that for almost all z_1 and z_2 in some small neighborhood in $\mathbb{R}^2$, the inequality

$$\left(F\left(z_1\right) - F\left(z_2\right)\right)^T \left(z_1 - z_2\right) \leq K\|z_1 - z_2\|^2 \tag{10}$$

holds for some constant K (see inequality (51) p. 218 of [12]). Since F is Lipschitz away from the discontinuity boundary, one easily checks that this condition is satisfied in this region.

We now consider initial value problems with initial conditions in the discontinuity boundary. There are two such discontinuity boundaries: the lines $y = 0$ and $y = 1$. First, let z be a point on the line $y = 1$. Let δ be small enough that $B_\delta(z)$ is a neighborhood contained within the set $\{(x,y) : y > 0\}$. Suppose that $z_1 = (x_1, y_1)$ satisfies $0 < y_1 < 1$ and $z_2 = (x_2, y_2)$ satisfies $y_2 > 1$. We consider $\left(F(z_2) - F(z_1)\right)^T \left(z_2 - z_1\right)$ for the following two cases:

1. If $H(z_1) \geq 0$, then $\left(F(z_2) - F(z_1)\right)^T \left(z_2 - z_1\right) = \left(0, -|H(z_2)| - H(z_1)\right)^T \left(x_2 - x_1, y_2 - y_1\right) < 0$, since $-|H(z_2)| - H(z_1) < 0$ and $y_2 - y_1 > 0$. Then, inequality (10) is satisfied with $K = 1$.

2. If $H(z_1) < 0$, then $F(z_2) - F(z_1) = (0, -|H(z_2)| + |H(z_1)|)$. The second entry may be estimated further: $-|H(z_2)| + |H(z_1)| \leq |H(z_1) - H(z_2)| \leq k_H\|z_1 - z_2\|$ where k_H is the Lipschitz constant of the function H. Therefore, the inequality (10) is satisfied with $K = k_H$.

A similar argument holds for the other discontinuity boundary, and therefore, system (8)-(9) satisfies the one-sided Lipschitz continuity condition (10), guaranteeing forward-time uniqueness of Filippov solutions.

Filippov theory further guarantees that a sliding solution exists at the discontinuity boundary. With the normal vector taken in the direction of increasing y, one checks that system (8)-(9) satisfies the conditions of Lemma 3 in [12]. This gives the sliding solution on the upper boundary $y = 1$. If $z(t)$ denotes this solution, then, with $\alpha = 1/2$,

$$\begin{aligned} \frac{dz}{dt} &= [G(x,1), \alpha(-|H|) + (1-\alpha)H] \\ &= [G(x,1), 0]. \end{aligned}$$

Furthermore, above the upper discontinuity boundary the vector field points downward, while below the lower boundary, the vector fields points up. Therefore, since any solution can only cross into or slide along the boundary, the strip is forward invariant. □

Armed with this result, we return to system (6). Recall that g and h are both C^1 functions over the plane. Henceforth, we consider the initial value problem in the state space $(A, \eta) \in \mathbb{R} \times \mathbb{R}$ endowed with the following piecewise Lipschitz vector field:

$$\dot{A} = g(A, \eta) \tag{11}$$

$$\dot{\eta} = \begin{cases} -|h| & \text{when } \eta > 1 \\ \frac{h-|h|}{2} & \text{when } \eta = 1 \\ h & \text{when } 0 < \eta < 1 \\ \frac{h+|h|}{2} & \text{when } \eta = 0 \\ |h| & \text{when } \eta < 0. \end{cases} \tag{12}$$

having an initial value $(A(0), \eta(0)) \in \mathbb{R} \times \mathbb{R}$.

An application of Theorem 3.1 shows that a unique forward-time Filippov solution to the initial value problem (11)-(12) exists. Furthermore, the forward invariance of the strip $\mathbb{R} \times [0,1]$ ensures the physically relevant aspect of the ice line η. We have arrived at the following conclusion:

Corollary 3.2. *There exists a unique Filippov solution to* (11)-(12)*. Furthermore, the strip* $\mathbb{R} \times [0,1]$ *is forward time invariant.*

Remark 3.3. *There are many possible ways of extending system* (6) *beyond the strip* $\mathbb{R} \times [0,1]$*. The choice of extension as done in system* (11)-(12) *assures the following points:*

1. *The attracting nature of the strip* $\mathbb{R} \times [0,1]$.
2. *The switching from sliding to crossing is entirely determined by zeroes of* $h(A, \eta)$*. This point is especially important in the modeling of the physical system, since* h *signals the ice line's advance or retreat and* h *should be "off" for some positive amount of time at the extreme ice line locations (pole or equator) and should "turn back on" when the system has reduced or increased enough greenhouse gases.*
3. *The time derivative of the sliding mode is entirely determined by the governing function of the greenhouse gas effect,* $g(A, \eta)$*. Again, this aspect is especially important in the modeling of the physical system, since at the extreme ice line location (pole or equator), the ice-albedo feedback shuts off and the greenhouse gas effect is the only driver of the system.*

3.3 Dynamics of the model

We now focus our analysis on the system (11)-(12). We show that its sliding dynamics guarantee exit from the snowball scenario, and use the Filippov framework to simulate stable small ice cap states and large amplitude periodic orbits.

The η-nullcline, given by $h(A, \eta) = 0$, is cubic in η and linear in A (see Table 1). It has a single fold in the region $0 \le \eta \le 1$ at $\eta = \eta_f \approx 0.77$, as can be seen in Figure 2. The A-nullcline is the horizontal line $\eta = \eta_c$. If $0 < \eta_c < 1$, the system has a single fixed point. The eigenvalues associated with this fixed point are

$$\lambda_{\pm} = \frac{1}{2}\left(\frac{\partial h}{\partial \eta} \pm \sqrt{\left(\frac{\partial h}{\partial \eta}\right)^2 - \frac{4\delta}{B}}\right).$$

From this and Figure 2, we see that if the fixed point lies above the fold and in the physical region, i.e. $\eta_f < \eta_c < 1$, then it is stable, and it is unstable when $0 < \eta_c < \eta_f$. This is reminiscent of the "slope-stability theorem" from the climate literature [4]. In that work, the authors related changes in slopes of equilibrium curves to changes in stability of equilibria for a globally averaged energy balance model. They found that a small ice cap or ice-free solution could be stable, a large ice cap was unstable, and a snowball state was again stable.

Remark 3.4. *In the case that* $\eta_c = 0, 1$*, one of the physical boundaries is entirely composed of equilibria, and stability is determined by the direction of the vector field near it. Physically, this degenerate scenario comes about when there is an absence of volcanism in the ice-covered state or a perfect balance between weathering and volcanism in the ice-free state. We do not discuss these cases further.*

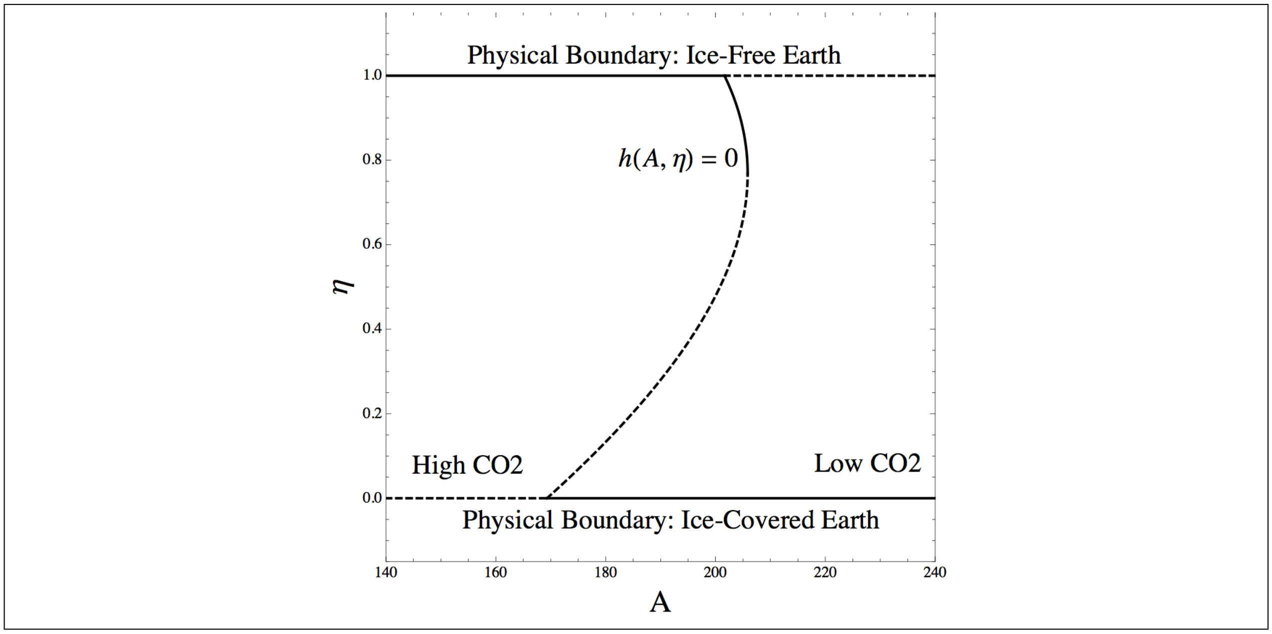


Figure 2: The physical region of the phase space and possible fixed points of the system given by the η-nullcline, $h = 0$. The location of the equilibrium is determined by the critical effective area of exposed land $0 < \eta_c < 1$. Solid black portions of the curve represent stable equilibria while the dashed lines denote unstable equilibria. Solid black portions of the boundary are attractive sliding regions and dashed boundaries are crossing regions.

3.3.1 Sliding and escape from Snowball Earth

Recall that η_c was defined to be a ratio of volcanism to weathering. We now prove that the system always recovers from the ice-covered regime provided this parameter is positive, i.e. there is some contribution from volcanic outgassing to atmospheric greenhouse gas content.

Theorem 3.5. *Suppose $\eta_c > 0$. Then any orbit of the system* (11)-(12) *that enters the boundary $\eta = 0$ must exit in finite time.*

Proof. Let A_* be the A-coordinate of the intersection of the η-nullcline with the lower ice boundary $\eta = 0$. Suppose an orbit enters the boundary $\eta = 0$. At such an occurrence $A \geq A_*$, since this is where $\dot{\eta} = h(A, \eta) \leq 0$. For $A \geq A_*$, the confinement vector field in the non-physical region $\eta < 0$ is given by $\dot{\eta} = |h(A, \eta)| \geq 0$. A is decreasing on either side of $\eta = 0$. The Filippov convention induces sliding according to

$$\begin{aligned}\dot{\eta} &= \frac{h(A,0) + |h(A,0)|}{2} = 0 \\ \dot{A} &= -\delta\eta_c.\end{aligned}$$

At the intersection point $A = A_*$, the vector field is tangent to the boundary and sliding terminates. For $A < A_*$ the boundary is a crossing region, and the vector fields on either side of it are identical with $\dot{\eta} > 0$. Moreover, there is a single orbit of the smooth system that passes through the tangency point. Due to forward-time uniqueness from Theorem 3.1, the sliding solution must follow this orbit and reenter the physical region.

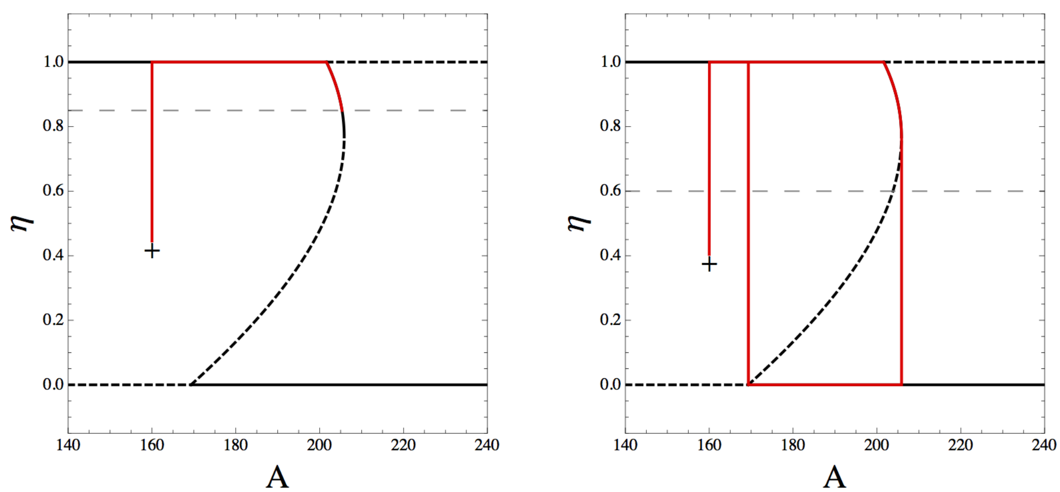
□

Remark 3.6. *A similar argument shows that any orbit that enters an ice-free state must exit in finite time. The required condition is that there is not a perfect balance between volcanism and weathering, i.e.* $\eta_c \neq 1$.

3.3.2 Numerical simulations

The Filippov framework also allows for numerical simulation of the model. Let (A_c, η_c) be the location of the single fixed point. We find that small ice cap states corresponding to fixed points with $\eta_f < \eta_c < 1$ are possible attractors of the system, as are periodic orbits born from the Hopf bifurcation at $\eta_c = \eta_f$.

The fold of the η-nullcline is a *canard point*, and the mechanism that produces the large amplitude periodic orbit in Figure 3 is a *canard explosion* [2, 9, 19]. While mathematically interesting, canard orbits are unlikely observables for planar systems because they occur within an exponentially small parameter range. However, they can be robust in higher dimensions which makes more complex oscillatory behavior possible, see e.g. [1, 8, 27, 30]. With this in mind, we remark that an interesting avenue for future research involves studying an extended version of the system (11)-(12) with an additional variable (w in [21]) and foregoing their invariant manifold reduction.



(a) Small ice cap equilibrium (b) Periodic orbit

Figure 3: Attractors of the system when (a) $\eta_c = 0.85$ and (b) $\eta_c = 0.6$. The $+$ symbol marks the intial condition and the horizontal long-dashed line is the A-nullcline. In (a), the orbit reaches the ice-free state and slides until it reaches the intersection of the folded curve $h(A, \eta) = 0$ with this boundary. It then enters the physical region and approaches the small ice cap equilibrium. In (b), the fixed point is unstable and the orbit oscillates between the ice-free and ice-covered boundaries. Simulations were performed using Mathematica 9.

3.4 Application to the Jormungand Model

The previous occurrence of a complete snowball event is still a highly contested topic in geology. In the model (11)-(12), the only possible stable fixed point is a small ice cap. In this section, we present a variant of the system, as introduced by Abbot, Voigt, and Koll [7], and employ the Filippov framework as above. By modifying the albedo function $\alpha(\eta, y)$ so that it differentiates between the albedo of bare ice and snow-covered ice, the new system has an additional stable state that corresponds to a large ice cap. In [7], this was called the *Jormungand* state because it allows for a snake-like band of open ocean at the equator. Moreover, there are again attracting periodic orbits. These can be seen in Figure 4.

In this version of the model, the albedo is defined as follows:

$$\alpha_J(\eta, y) = \begin{cases} \alpha_w, & y < \eta \\ \frac{1}{2}(\alpha_w + \alpha_2(\eta)), & y = \eta \\ \alpha_2(y), & y > \eta, \end{cases} \tag{14}$$

where $\alpha_2(y) = \frac{1}{2}(\alpha_s + \alpha_i) + \frac{1}{2}(\alpha_s - \alpha_i) \tanh M(y - 0.35)$.

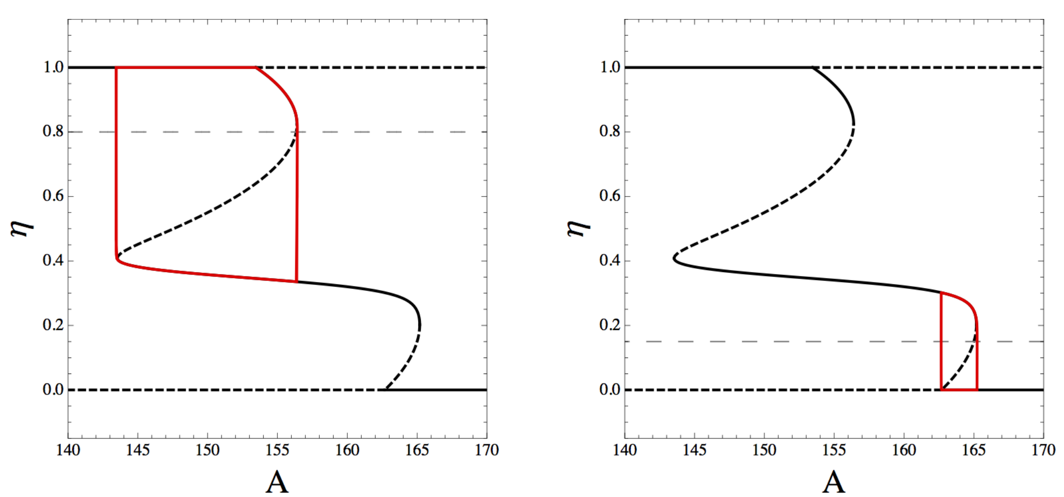
Here α_w is the albedo of open water, α_i is the albedo of bare sea ice, and α_s is the albedo of snow-covered ice. The model assumes sea ice aquires a snow cover only for latitudes above $y = 0.35$. We modify $h(A, \eta)$ by replacing α with α_J:

$$h_J(A, \eta) = \frac{Q}{B + C}\left(s(\eta)(1 - \alpha_J(\eta, \eta)) + \frac{C}{B}(1 - \overline{\alpha_J}(\eta))\right) - \frac{A}{B} - T_c \tag{13}$$

where we note that although $\alpha(\eta, y)$ has a discontinuity when $y = \eta$, both $\alpha(\eta, \eta)$ and $\overline{\alpha_J}(\eta) = \int_0^1 \alpha(\eta, y)s(y)dy$ are smooth functions of η.

For this system, we can immediately apply Theorems 3.1 and 3.5 to obtain the following result.

Corollary 3.7. *The physical region is forward invariant with respect to the system* (11)-(12) *where* h *is replaced by* h_J*. Solutions exist and are unique in forward time, and any trajectory that enters an ice-covered state must exit in finite time provided* $\eta_c \neq 0$*.*



(a) (b)

Figure 4: Periodic orbits of the Jormungand system when (a) $\eta_c = 0.8$ and (b) $\eta_c = 0.15$. The folded curve is the η-nullcline $h_J(A, \eta) = 0$ and dashing is as in Figures 2 and 3.

4 Discussion

In this article we have extended a class of energy balance models to include a greenhouse gas component. The resulting system was nonsmooth due to physical constraints and did not

Parameters	Value	Units
T_c	0	°C
M	25	dimensionless
α_w	0.35	dimensionless
α_i	0.45	dimensionless
α_s	0.8	dimensionless

Table 2: Parameter values as in Table 1 unless specified above. Additional values taken from [7].

immediately fit into an existing analytical framework. However, we defined an extended system that allowed us to utilize the theory of Filippov and obtain the expected motion, in agreement with physical arguments from climate literature. More specifically, we proved existence and uniqueness of forward-time solutions and showed that the system was confined to the physical region with sliding on the ice-covered and ice-free boundaries. We showed that the extended system always escapes from the ice-covered scenario, thus supporting Kirschvink's hypothesis about carbon dioxide accumulation in a snowball scenario due to a shut down of chemical weathering processes [18]. Using our extended system, we found that small ice-cap and large amplitude ice-cover oscillations were possible attractors of the system. We then applied our results to the case of Jormungand world and showed that it is possible to obtain further oscillatory dynamics between no ice cover, large ice-cover, and full ice-cover states.

General mathematical questions brought about by this work have to do with building a general framework for such models so that there is no need to appeal to artificial tools. A first step toward dealing with physical boundaries might be a general theory for semiflows generated by smooth vector fields on manifolds with boundary. In addition, the stable portions of the physical boundaries in our model should be thought of as equilibria of the fast subsystem, i.e. as part of the *critical manifold* from geometric singular perturbation theory [17]. In fact, they are much more *normally hyperbolic* than the curve $h = 0$; they are reachable in finite time! However, the current theory cannot be directly applied to the full discontinuous system and we remark that developing an analogous Fenichel theory [11] for singularly perturbed discontinuous systems with sliding is an interesting avenue for future research.

Another interesting future direction is to study the explicit effect of the temperature (here we have considered it only through the equation for η based on the reduction by McGehee and Widiasih [21]) on the dynamics of the system. In a nonsmooth system, the addition of a stable dimension may destroy an existing stable periodic orbit [26]. Moreover, an additional dimension could result in more interesting glacial dynamics, such as mixed mode oscillations.

Finally, it is natural to ask how the shape of the η-nullcline changes with physical parameters and how this affects system dynamics, e.g. could the lower fold in Figure (4) move to the left of the upper fold? We refer the reader to [29] for a number of examples.

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