Set 6 Solutions

Work Exercise 2, Section 6.5, from the textbook, repeated here for your convenience.

2. Consider the function \( \phi : f \mapsto \phi(f) \) defined in Eq. (6.9), where \( \delta = \frac{1}{6} \) and \( R = \frac{3}{2} \).

   (i) The equation \( \lambda f = \phi(f) \) has exactly one solution \( f = f^* < 0 \) if \( \lambda = \frac{4}{5} \). Find this solution.

   (ii) How many negative solutions does the equation \( \lambda f = \phi(f) \) have if \( \lambda \in (0, \frac{4}{5}) \)? Where are these solutions relative to the solution found in (i)?

   (iii) Draw a bifurcation diagram showing the negative solutions of \( \lambda f = \phi(f) \) on the vertical axis and the parameter \( \lambda > 0 \) on the horizontal axis.

2(i) Equation (6.9) is

\[
\phi(f^*; R, \delta) = \frac{\delta R}{\delta + |f^*|} - \frac{1}{1 + |f^*|}.
\]

We are looking for a negative solution to \(-\lambda f^* = \phi(f^*)\) for the parameter values \( \delta = \frac{1}{6} \) and \( R = \frac{3}{2} \). To get a feeling for the question, we reproduce Figure 6.1 in the textbook for these values.

It looks like there is a tangent between the line and the curve at about \( f = -0.25 \). In search of this solution, we let \( \xi = -f^* > 0 \) and write Equation (6.9) as
-\lambda \xi = \frac{\delta R}{\delta + \xi} - \frac{1}{1 + \xi}

Convert this equation to a cubic polynomial by clearing the denominators:

\[ -\lambda (\delta + (1 + \delta)\xi^2 + \xi^2) = \delta R(1 + \xi) - \delta - \xi \]

\[ \lambda \xi^3 + \lambda (1 + \delta)\xi^2 + \lambda \delta \xi + \delta R\xi - \xi + \delta R - \delta = 0 \]

We now substitute the values \( \delta = \frac{1}{6}, R = \frac{1}{2}, \) and \( \lambda = \frac{5}{6}: \)

\[
\begin{align*}
\lambda \xi^3 + \lambda (1 + \delta)\xi^2 + \lambda \delta \xi + \delta R\xi - \xi + \delta R - \delta &= 0 \\
4\xi^3 + 4(1 + \frac{1}{6})\xi^2 + \frac{4}{5} + \frac{1}{6} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{6} &= 0 \\
\frac{4}{5}\xi^3 + \frac{14}{15}\xi^2 - \frac{37}{60} \xi + \frac{1}{12} &= 0 \\
48\xi^3 + 56\xi^2 - 37\xi + 5 &= 0 \quad (*)
\end{align*}
\]

We are looking for a double root, which occurs when the derivative is zero:

\[
144\xi^2 + 112\xi - 37 = 0 \\
(4\xi - 1)(36\xi + 37) = 0
\]

We see that the derivative is zero at \( \xi = \frac{1}{4}, \) which is a candidate for a double root.

Substituting this value into equation (*) we see that it is indeed a root:

\[
48\left(\frac{1}{4}\right)^3 + 56\left(\frac{1}{4}\right)^2 - 37\left(\frac{1}{4}\right) + 5 = 0,
\]

leading us to factor equation (*):

\[
48\xi^3 + 56\xi^2 - 37\xi + 5 = (4\xi - 1)^2 (12\xi + 5) = 0,
\]

which implies that \( \xi = \frac{1}{4} \) is the only positive root and hence that \( f^* = -\frac{1}{4} \) is the only negative solution to equation (6.9).

2(ii) From the above figure it is clear that decreasing the slope of the line \( \lambda f' \) creates two negative solutions, one on either side of \( f' = 1/4. \) Thus, for \( 0 < \lambda < 4/5, \) we have two negative solutions \( f_1^* \) and \( f_2^*, \) with \( f_1^* < 1/4 < f_2^* \) and with \( f_1^* \) and \( f_2^* \) both approaching \( 1/4 \) as \( \lambda \to 4/5. \)
2(iii) To produce the requested bifurcation diagram, we solve Equation (6.9) for $\lambda$ as a function of $f$:

$$\lambda = \frac{\phi(f)}{f} = \frac{1}{f} \left( \frac{\delta R}{\delta + |f|} - \frac{1}{1+|f|} \right)$$

![Bifurcation diagram](image-url)