Math 5490 11/10/2014

Topics in Applied Mathematics: Introduction to the Mathematics of Climate
Mondays and Wednesdays 2:30 – 3:45
http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

Streaming video is available at http://www.ima.umn.edu/videos/
Click on the link: "Live Streaming from 305 Lind Hall".

Participation: https://umconnect.umn.edu/mathclimate

Dynamical Systems
Nonlinear Systems

If \( \xi \) is small, i.e., if \( \xi \) is close to \( \xi_0 \), then solutions of \( \xi' = f(\xi) \) are close to solutions of \( \xi' = f(\xi_0) \).

\[
\frac{d\xi}{dt} \approx f(\xi) + Df(\xi_0)(\xi - \xi_0)
\]

Rest point \( \xi_0 \) (0)

Introduce \( Df(\xi_0) \).

Then \( \xi' = f(\xi) \) (0) (\( \xi_0 \) is asymptotically stable for \( \xi' = f(\xi_0) \).

Basic Idea

Linear approximation:

\[
\xi' \approx \xi'_{\text{linear}} = \xi'_{\text{0}} + Df(\xi_0)(\xi - \xi_0)
\]

In particular, the rest point \( \xi_0 \) is asymptotically stable for \( \xi' = f(\xi_0) \).

Dynamical Systems
Flows
"Vector Fields Determine Flows"

A flow is a continuous map \( \phi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfying

\[
\phi(t, x_0) = x(t, x_0)
\]

for all \( t, x_0 \). Alternate Notation

\[
\phi(t, x_0) = x(t, x_0)
\]

Group Property

\( \phi(t_1 + t_2, x_0) = \phi(t_1, \phi(t_2, x_0)) \)

Smooth: \( C^n, n > 0 \)

(n times continuously differentiable)

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Check properties:

\[
\phi(t_0, 0) = e^{t_0} \phi(0, 0) = 0
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Example

vector field: \( \dot{x} = 2x, \ x \in \mathbb{R} \) initial value: \( x(0) = x_0 \)

solution: \( x = e^{2t} x_0 \)

flow:

\[
\phi(t, 0) = e^{t} \phi(0, 0) = x_0
\]

Check properties:

\[
\phi(t_0, 0) = e^{t_0} \phi(0, 0) = x_0
\]

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\phi(t_1, \phi(t_0, 0)) = \phi(t_1 + t_0) = e^{t_1} e^{t_0} \phi(0, 0) = x_0
\]

Example

vector field: \( \dot{x} = x^2, \ x \in \mathbb{R} \) initial value: \( x(0) = x_0 \)

solution:

\[
\frac{dx}{dt} = x^2 \Rightarrow x^2 \, dt = dx \Rightarrow \int x^2 \, dt = \int dx \Rightarrow \frac{1}{3} x^3 = t + c \Rightarrow x = \sqrt[3]{3(t + c)}
\]

flow:

\[
\phi(t, 0) = x_0
\]
Dynamical Systems

Flows

"Vector Fields Determine Flows"

Example

vector field: \( \dot{x} = x^2 \), \( x \in \mathbb{R} \)
initial value: \( x(0) = x_0 \)
flow:

\[
\phi(x_0,t) = \frac{x_0}{1 - x_0^2 t}
\]
group property

Check properties:

\[
\phi(\phi(x_0,t),t) = \phi(x_0,t) \quad \Rightarrow \quad \phi(\phi(x_0,t),t) = \frac{x_0}{1 - x_0^2 (1 - x_0^2 t)} = \frac{x_0}{1 - x_0^2 t}
\]

local flow:

Solutions exist for some time interval.

Example:

vector field: \( \dot{x} = x^2 \), \( x \in \mathbb{R} \)
initial value: \( x(0) = x_0 \)
flow:

\[
\phi(x_0,t) = \frac{x_0}{1 - x_0^2 t}
\]

Issue:
Solutions do not exist for all time.

\[
\phi(x_0,t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \frac{1}{x_0^2}
\]

Example:

vector field: \( \dot{x} = x^2 \), \( x \in \mathbb{R} \)
initial value: \( x(0) = x_0 \)
flow:

\[
\phi(x_0,t) = \frac{x_0}{1 - x_0^2 t}
\]

Group Property

\[
\phi(x_0 + x_1, t) = \frac{x_0 + x_1}{1 + x_0 x_1 t}
\]

equal

Example:

vector field: \( \dot{x} = x^2 \), \( x \in \mathbb{R} \)
initial value: \( x(0) = x_0 \)
flow:

\[
\phi(x_0,t) = \frac{x_0}{1 - x_0^2 t}
\]

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\]

Going back in time is the same as following the negative of the vector field.
Dynamical Systems

"Vector Fields Determine Flows"

Example

vector field: \[ x = 1 - e^x, \quad x \in \mathbb{R} \]
local flow: \[ \phi(x,t) = \frac{x + \tanh(t)}{1 + x \tanh(t)} \]

vector field: \[ x = -\left(1 - e^x\right), \quad x \in \mathbb{R} \]
local flow: \[ \psi(x,t) = \frac{x + \tanh(-t)}{1 + x \tanh(-t)} \]

\[ \psi(x,t) = \frac{x + \tanh(-t)}{1 + x \tanh(-t)} = \phi(x,-t) \]

Nonlinear Systems

Rest point \( p \) : \( f(p) = 0 \)

Stability

Lyapunov stable
asymptotically stable
unstable

What about saddles?

If one of the eigenvalues of the Jacobian matrix is positive and the other is negative, then there are two smooth curves intersecting at \( p \), \( W^u(p) \) (the unstable manifold), and \( W^s(p) \) (the stable manifold) satisfying these properties:

- \( x \in W^u(p) \) \( \Rightarrow \phi(x,t) \to p \) as \( t \to \infty \)
- \( x \in W^s(p) \) \( \Rightarrow \psi(x,t) \to p \) as \( t \to -\infty \)

\( W^u(p) \) is tangent at \( p \) to the eigenvector corresponding to the negative eigenvalue.
\( W^s(p) \) is tangent at \( p \) to the eigenvector corresponding to the positive eigenvalue.
Nonlinear Systems

Rest point: \((0,0)\), \((-1,0)\), \((1,0)\)

Example:
\[
\begin{align*}
\frac{dx}{dt} &= x - x^3 + y \\
\frac{dy}{dt} &= -y
\end{align*}
\]

\[DF(x,y) = \begin{bmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{bmatrix}\]

\[DF(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad DF(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad DF(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}\]

Sinks: eigenvalues: \(-2\), \(-1\) 
Eigenvectors: \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\)

Saddle: eigenvalues: \(-1\), \(-1\) 
Eigenvectors: \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\)

Stable manifold
Unstable manifold

What about saddles?

Dynamical Systems
Example

\[
\begin{align*}
\frac{dx}{dt} &= x - x^2 + y \\
\frac{dy}{dt} &= -y
\end{align*}
\]

Dynamical Systems
Nonlinear Systems

Example

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix}
= \begin{pmatrix}
1 & -1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\text{stable node} \quad A = Df(1,0) - Df'(1,0) = \begin{pmatrix}
-2 & 1 \\
0 & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
eigenvalues \\
eigenvectors
\end{pmatrix}
= \begin{pmatrix}
2 & 1 \\
0 & 1
\end{pmatrix}
\]

eigenvalues: \(2, -1\)

eigenvectors: \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\)

Dynamical Systems
Nonlinear Systems

Example

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Dynamical Systems
Nonlinear Systems

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Dynamical Systems
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Dynamical Systems
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Dynamical Systems
Nonlinear Systems

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Dynamical Systems
Stommel's Model

Stable spiral
Saddle
Stable node

Stommel, TELLUS XII (1961)