**Math 5490**

Topics in Applied Mathematics: Introduction to the Mathematics of Climate

Mondays and Wednesdays 2:30 – 3:45

http://www.math.umn.edu/~mcgehee/teaching/Math5490-2014-2Fall/

Streaming video is available at http://www.ima.umn.edu/videos/

Click on the link: “Live Streaming from 305 Lind Hall”.

Participation: https://umconnect.umn.edu/mathclimate

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**Dynamical Systems**

**Bifurcation Theory**

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**Dynamical Systems**

Saddle-Node Bifurcation

1. Increase the flow resistance.
2. Not much different, but it is easier to get to $c$.

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**Dynamical Systems**

Saddle-Node Bifurcation

Increase the flow resistance. The saddle and the stable node start to merge.

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**Dynamical Systems**

Saddle-Node Bifurcation

Increase the flow resistance. The saddle and the stable node have disappeared. The Gulf Stream will eventually reverse.

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**Dynamical Systems**

Bifurcation Theory

Setup:

$x = f(x, \mu), \ x \in \mathbb{R}^n, \ \mu \in \mathbb{R}^m$

Rest point at $x = 0$ when $\mu = 0$: $f(0,0) = 0$

What happens when we change the parameters?
Dynamical Systems

Bifurcation Theory

\[ \dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m \]
rest point at \( x = 0 \) when \( \mu = 0 \): \( f(0,0) = 0 \)

No Bifurcation (Poincaré Continuation)

The Jacobian matrix \( Df(0,0) \) is nonsingular, i.e., has no zero eigenvalues.

Conclusion
For small values of \( \mu \), there is a rest point \( p(\mu) \) satisfying
\[ p(0) = 0, \quad f(p(\mu), \mu) = 0. \]
The rest point "continues" for small parameter values.

Poincaré Continuation

If Jacobian matrix \( Df(0,0) \) is nonsingular, then, for small values of \( \mu \), there is a rest point \( p(\mu) \) satisfying
\[ p(0) = 0, \quad f(p(\mu), \mu) = 0. \]

Idea of Proof

We can write
\[ f(x, \mu) = A x + B \mu + O^3(x, \mu) = 0, \]
where \( A = Df(0,0) \) and \( B \) is an \( n \times m \) matrix and solve for \( x \):
\[ x = p(\mu) = A^{-1}B + O^3(p(\mu)). \]

There's more!

If \( f \) is continuously differentiable (C\(^2\)), then the Jacobian matrix \( Df(p(\mu), \mu) \) varies continuously with \( \mu \), as do the eigenvalues and eigenvectors.

If the rest point at \( \mu = 0 \) is hyperbolic (or a saddle, or a stable node, or an unstable node, or a stable spiral, or an unstable spiral), then the rest point \( p(\mu) \) inherits the property for small values of \( \mu \).

Example

\[ \dot{x} = f(x, \mu) = x - 2x + x^2 \]
\[ Df(x, \mu) = -2 + 2x \]
so there is a rest point \( x = p(\mu) \) satisfying \( p(0) = 0 \).
\[ x = p(\mu) = \frac{2 - \sqrt{4 - 8\mu}}{4}. \]

For each value of \( \mu \) close to 0, there is a unique rest point near \( x = 0 \).
In this example, we can solve explicitly:
\[ x^2 + 2x - \mu = 0 \]
\[ x = -1 - \sqrt{1 + \mu} \]
Since \( p(0) = 0 \), we take the "+" sign:
\[ x = p(\mu) = -1 + \sqrt{1 + \mu}. \]

Kaper & Stryja, 2013

Example

\[ \dot{x} = f(x, \mu) = x - 2x + x^2 \]
\[ Df(x, \mu) = -2 + 2x \]
so the rest point \( x = p(\mu) \) has an eigenvalue near -2 for small \( \mu \) and hence is asymptotically stable.

Note that there is another rest point at \( x = -2 \) for \( \mu = 0 \).
Its eigenvalue is \( Df(-2,0) = -2 + 2(-2) = 0 \), so it is unstable.
Furthermore, for small values of \( \mu \), there is a unique rest point \( p(\mu) \) near \( x = -2 \), and that rest point is unstable.
Dynamical Systems

Classification

Determinant

Poincaré continuation falls when determinant = 0.

What happens when it fails?

Poincaré continuation

trace

determinant

Dynamical Systems

Bifurcation Theory

Example

\[ x = f(x, \mu) = \mu - 2x - x^2 \]

Two rest points: \( \mu - 2x - x^2 = 0 \)

rest point: \( p_1 = -1 \pm \sqrt{\mu} \) eigenvalue: \( 2 \sqrt{\mu} \)

rest point: \( p_2 = 1 - \sqrt{\mu} \) eigenvalue: \(-2 \sqrt{\mu}\)

When \( \mu = -1 \), the rest points merge, and the eigenvalue becomes 0.

The rest point becomes a “saddle-node”.

Kaper & Engler, 2013

Rest Points:

0: two rest points: \((x, y) = (\sqrt{\mu}, 0)\)

0: one rest point: \((x, y) = (0, 0)\)

0: no rest point

Jacobian \( D_f((x, y), \mu) = \begin{bmatrix} 2x & 0 \\ 0 & -2 \end{bmatrix} \)

\( \mu < 0 \)

\( \mu = 0 \)

\( \mu > 0 \)

The local structure is not determined by the linearized equations.
Dynamical Systems

Example
\[ x = \mu + x^2 \]
\[ y = -2y \]

Steady Points
\[ 3x - x^3 - \mu = 0, \quad x = 3x - x^3 \]

Stable: \[ D_f(x, \mu) < 0, \quad \text{if } \|x\| > 1 \]

Unstable: \[ D_f(x, \mu) > 0, \quad \text{if } \|x\| < 1 \]

\[ D_f(x, \mu) = 0, \quad \text{if } x = \pm 1 \]

Stommel Model

\[ x = \frac{1}{\beta} \]
\[ \mu = 2 \]
\[ \beta = 0.3 \]

Increase the flow resistance. The saddle and the stable node start to merge.

\[ x = \frac{1}{\beta} \]
\[ \mu = 2 \]
\[ \beta = 0.4 \]

Increase the flow resistance. The saddle and the stable node have disappeared. The Gulf Stream will eventually reverse.
Dynamical Systems

Bifurcation Theory

Example
\[ x = 3x - x^3 - \mu \]

Hysteresis

The system has a memory of where it has been. Returning parameters to the previous state might not return the system to the previous state.

Start here

undesirable

desirable

Hysteresis

Decrease the parameter to -2 (the tipping point).

undesirable
desirable

Hysteresis

Decrease to below the tipping point. The system flips to the other stable state.

undesirable
desirable

Hysteresis

We now increase the parameter back to its starting value, but the system stays in the new state.

undesirable
desirable

Hysteresis

We must increase the parameter back to the other tipping point (\( \mu = 2 \)) before we can return to the previous state.

undesirable
desirable

Hysteresis

We must increase the parameter back to the other tipping point (\( \mu = 2 \)) (and beyond) before we can return to the previous state.
Dynamical Systems
Bifurcation Theory

Example
\[ x = 3x - x^3 - \mu \]

Hysteresis
Now we can decrease the parameter back to its original value, returning the system to its original state.

Cusp Catastrophe
\[ \dot{x} = \lambda + \mu x - x^3 \]

rest points: \( \dot{\lambda} + \mu x - x^3 = 0 \)
saddle-nodes: \( \dot{f}(x, \lambda, \mu) = \mu - 3x^2 = 0 \)

Guckenheimer via Kaper & Engler, 2013

Kaper & Engler, 2013