A combinatorial model for highest weights of finite dimensional representations of $\mathfrak{gl}(m, n)$

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Facts about $\mathfrak{gl}(m, n)$:
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- $\mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,
Facts about $\mathfrak{gl}(m, n)$:

- $\mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,

- $\mathfrak{g}_0 = \begin{pmatrix} m \vert n \end{pmatrix}$;
  $\mathfrak{g}_1 = \begin{pmatrix} m \vert n \end{pmatrix}$.
Facts about $\mathfrak{gl}(m, n)$:

- $\mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_\bar{1}$,

- $\mathfrak{g}_0 = \begin{pmatrix} m & \ldots & n \\ m & \ldots & n \\ \vdots & \ddots & \vdots \\ m & \ldots & n \end{pmatrix}$;

- $\mathfrak{g}_\bar{1} = \begin{pmatrix} m & \ldots & n \\ m & \ldots & n \\ \vdots & \ddots & \vdots \\ m & \ldots & n \end{pmatrix}$

- $\mathfrak{h} = $ Cartan subalgebra
Facts about $\mathfrak{gl}(m, n)$:

- $\mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_\bar{1}$

- $\mathfrak{g}_0 = \begin{pmatrix} \mathbb{R}^n & 0 \\ 0 & \mathbb{R}^m \end{pmatrix}$; $\mathfrak{g}_\bar{1} = \begin{pmatrix} \mathbb{R}^m & 0 \\ 0 & \mathbb{R}^n \end{pmatrix}$

- $\mathfrak{h} = \text{Cartan subalgebra} = \text{diagonal matrices}$,
Facts about $\frak{gl}(m, n)$:

- $\frak{gl}(m|n) = \frak{g}_0 \oplus \frak{g}_1$,

- $\frak{g}_0 = m \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$; $\frak{g}_1 = n \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$

- $\frak{h} = \text{Cartan subalgebra} = \text{diagonal matrices}$,

- $\frak{h}^* = \text{span}(\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n)$,
Facts about $\mathfrak{gl}(m, n)$:

- $\mathfrak{gl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, 

\[
\begin{pmatrix}
\begin{array}{c|c}
\vline & \\
\hline & \\
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{c|c}
\vline & \\
\hline & \\
\end{array}
\end{pmatrix}
\]

- $\mathfrak{h} = \text{Cartan subalgebra} = \text{diagonal matrices}$,
- $\mathfrak{h}^* = \text{span}(\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n)$,
- Bilinear form: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$, $(\varepsilon_i, \delta_j) = 0$,
Facts about $\mathfrak{gl}(m, n)$:

- $\mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,

$$
\mathfrak{g}_0 = \begin{pmatrix}
\begin{array}{ccc}
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\end{array}
& 0 \\
0 & \begin{array}{ccc}
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\end{array}
& \\
\end{pmatrix},
\mathfrak{g}_1 = \begin{pmatrix}
\begin{array}{ccc}
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\end{array}
& 0 \\
0 & \begin{array}{ccc}
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\vline & \vline & \vline \\
\end{array}
& \\
\end{pmatrix}
$$

- $\mathfrak{h} = \text{Cartan subalgebra} = \text{diagonal matrices}$,
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- Bilinear form: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$, $(\varepsilon_i, \delta_j) = 0$,
- Weyl group: $S_m \times S_n$,
Facts about $\mathfrak{gl}(m, n)$:

- $\mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$
  
  $$
  \mathfrak{g}_0 = \begin{pmatrix}
  \begin{array}{ccc}
  & & \\
  & & \\
  & & \\
  \end{array}
  \\
  m & | & n \\
  & | & \\
  & | & \\
  0 & & \\
  0 & & \\
  0 & & \\
  \end{pmatrix}; \\
  \mathfrak{g}_1 = \begin{pmatrix}
  \begin{array}{ccc}
  & & \\
  & & \\
  & & \\
  \end{array}
  \\
  m & | & n \\
  & | & \\
  & | & \\
  0 & & \\
  0 & & \\
  0 & & \\
  \end{pmatrix}
  $$

- $\mathfrak{h} = \text{Cartan subalgebra} = \text{diagonal matrices}$,
- $\mathfrak{h}^* = \text{span}(\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n)$,
- Bilinear form: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$, $(\varepsilon_i, \delta_j) = 0$,
- Weyl group: $S_m \times S_n$,
- Roots: $\{\varepsilon_i - \varepsilon_j\}$, $\{\delta_i - \delta_j\}$; $\{\pm(\varepsilon_i - \delta_j)\}$,
Facts about \( \mathfrak{gl}(m, n) \):

- \( \mathfrak{gl}(m|n) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \),
- \( \mathfrak{g}_0 = \begin{pmatrix} m & \vline & n \\ \hline \multirow{m}{*}{\text{\vdots}} & \vline & \multirow{m}{*}{0} \\ \hline \multirow{m}{*}{0} & \vline & \multirow{m}{*}{\text{\vdots}} \\ \hline \end{pmatrix} \);
- \( \mathfrak{g}_1 = \begin{pmatrix} m & \vline & n \\ \hline \multirow{m}{*}{0} & \vline & \multirow{m}{*}{\text{\vdots}} \\ \hline \multirow{m}{*}{\text{\vdots}} & \vline & \multirow{m}{*}{\text{\vdots}} \\ \hline \end{pmatrix} \);
- \( \mathfrak{h} = \text{Cartan subalgebra} = \text{diagonal matrices}, \)
- \( \mathfrak{h}^* = \text{span}(\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n), \)
- Bilinear form: \( (\varepsilon_i, \varepsilon_j) = \delta_{ij}, (\delta_i, \delta_j) = -\delta_{ij}, (\varepsilon_i, \delta_j) = 0, \)
- Weyl group: \( S_m \times S_n, \)
- Roots: \( \{\varepsilon_i - \varepsilon_j\}, \{\delta_i - \delta_j\}; \{\pm(\varepsilon_i - \delta_j)\}, \)
- Choice of simple roots \( \Leftrightarrow \) ordering of \( \{\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n\}. \)
A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$
A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$

Arc diagram:

Sequence of $m \bullet$’s and $n \times$’s encodes the choice of base (simple roots).
Combinatorial model

Arc diagram:

●  X  X  ●  X  ●  X  ●  X  ●  X

11  1  5  6  6  6  6  6  2  13
Arc diagram:

\[ \bullet \times \times \bullet \times \bullet \times \bullet \times \bullet \times \]

\[
\begin{array}{cccccccccccc}
11 & 1 & 5 & 6 & 6 & 6 & 6 & 2 & 13 \\
\end{array}
\]

Integers (call them \textit{entries}) under the nodes give coordinates of the weight

\[
\lambda + \rho = 11\epsilon_1 + 6\epsilon_2 + 6\epsilon_3 + 2\epsilon_4 - \delta_1 - 5\delta_2 - 6\delta_3 - 6\delta_4 - 13\delta_5
\]
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Arc diagram:

Arcs denote a choice of \((\lambda + \rho)\)-maximal isotropic subset:

\[
S = \{ \varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3 \}.
\]
Arc diagram:

```
  ●   X   X   ●   X   ●   X   ●
  11  1   5   6   6   6   2   13
```

Arcs denote a choice of "$(\lambda + \rho)$-maximal isotropic subset":

$$S = \{ \varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3 \}.$$ 

Combinatorial conditions:
Arc diagram:

![Arc diagram](image)

Arcs denote a choice of "\((\lambda + \rho)\)-maximal isotropic subset":

\[ S = \{\varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3\}. \]

Combinatorial conditions:
- \( S \) is a collection of arcs with pairwise disjoint supports,
A combinatorial model for highest weights of f.d. modules of \( \mathfrak{g}l(m,n) \)

Arc diagram:

\[
\begin{array}{cccccccc}
\bullet & \times & \times & \bullet & \times & \bullet & \times & \bullet \\
11 & 1 & 5 & 6 & 6 & 6 & 6 & 2 & 13
\end{array}
\]

Arcs denote a choice of "\((\lambda + \rho)\)-maximal isotropic subset" :

\[
S = \{ \varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3 \}.
\]

Combinatorial conditions:
- \( S \) is a collection of arcs with pairwise disjoint supports,
- each arc has one end over a \( \times \) and the other over a \( \bullet \),
Arc diagram:

\[
\begin{array}{cccccccc}
\bullet & \times & \times & \bullet & \times & \bullet & \times & \bullet \\
11 & 1 & 5 & 6 & 6 & 6 & 6 & 2 & 13
\end{array}
\]

Arcs denote a choice of “\((\lambda + \rho)\)-maximal isotropic subset”:

\[S = \{\varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3\}.\]

Combinatorial conditions:

- \(S\) is a collection of arcs with pairwise disjoint supports,
- each arc has one end over a \(\times\) and the other over a \(\bullet\),
- the entries corresponding to the ends are equal,
A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$

Arc diagram:

\[ \bullet \times \times \bullet \times \times \bullet \times \bullet \times \bullet \times \times \bullet \times \]

\[ 11 \quad 1 \quad 5 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 2 \quad 13 \]

Arcs denote a choice of “$(\lambda + \rho)$-maximal isotropic subset”:

\[ S = \{ \varepsilon_2 - \delta_4, \delta_3 - \varepsilon_3 \}. \]

Combinatorial conditions:
- $S$ is a collection of arcs with pairwise disjoint supports,
- each arc has one end over a $\times$ and the other over a $\bullet$,
- the entries corresponding to the ends are equal,
- $S$ is maximal with respect to the above properties.
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A combinatorial model for highest weights of f.d. modules of \( \mathfrak{gl}(m, n) \)
A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m,n)$
Combinatorial model

Arc diagram:

11 1 5 6 6 6 6 2 13

Moves:
- Change $S$,
Combinatorial model

Arc diagram:

Moves:

- Change $S$,
- $a \quad \times \quad b \quad \leftrightarrow \quad \times \quad \bullet$
A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$
A combinatorial model for highest weights of f.d. modules of \( \mathfrak{gl}(m, n) \)

**Arc diagram:**

Moves:
- Change \( S \),
- \( \begin{array}{c} \bullet \\ a \end{array} \rightarrow \begin{array}{c} \times \\ b \end{array} \rightarrow \begin{array}{c} \times \\ a \end{array} \) \( \begin{array}{c} \times \\ b \end{array} \rightarrow \begin{array}{c} \bullet \\ a \end{array} \),
- \( \begin{array}{c} \bullet \\ a \\ a \end{array} \rightarrow \begin{array}{c} \times \\ a + 1 \end{array} \rightarrow \begin{array}{c} \times \\ a + 1 \end{array} \).  

Remark: The size of \( S \) is independent of the base.
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Arc diagram:

Moves:
- Change $S$, $a b \leftrightarrow b a$, $a a \leftrightarrow a + 1 a + 1$

Condition ("dominance"): In any base,

In any base,
A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$

Arc diagram:

Moves:
- Change $S$,
- $a \leftrightarrow b$
- $a \leftrightarrow a + 1$

Condition ("dominance"):
In any base,
- $a \prec b$,
- $a \succ b$.
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Arc diagram:

Moves:

- Change $S$,
- $a \hspace{1em} b \leftrightarrow b \hspace{1em} a$,
- $a \hspace{1em} a \leftrightarrow a + 1 \hspace{1em} a + 1$.

Condition ("dominance"): In any base,

- $a \hspace{1em} b \Rightarrow a > b$,
- $a \hspace{1em} b \Rightarrow a < b$. 
Example 1: Shortening

Motivation (conj. Kac-Wakimoto, 1994): if module has arc diagram with only short arcs (in some base) then it is “tame,” i.e. it has a nice character formula.
Example 1: Shortening

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Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base \(\iff\) it is \textit{totally connected} in the standard base.
Example 1: Shortening

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Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is totally connected in the standard base.

Totally connected:
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

Totally connected:

Not totally connected:
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\iff$)

![Diagram of arcs and dots with numbers](image-url)
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is totally connected in the standard base.

“Proof:” (⇐)

\[
\begin{array}{ccccccccc}
\bullet & \times & \times & \times & \times & \times & \times & \times & \bullet \\
9 & 8 & 6 & 5 & 3 & 1 & 5 & 7 & 9 & 17
\end{array}
\]
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base \(\iff\) it is \textit{totally connected} in the standard base.

“Proof:” \((\Leftarrow)\)
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is totally connected in the standard base.

“Proof:” (⇐)
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

"Proof:" $(\Leftarrow)$
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\Leftrightarrow$ it is *totally connected* in the standard base.

“Proof:” ($\Leftarrow$)

A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is totally connected in the standard base.

“Proof:” (⇔)

\[
\begin{align*}
\text{\textbullet} & \quad \times & \quad \rightarrow & \quad \times & \quad \text{\textbullet} \\
3 & \quad 9 & \quad 8 & \quad 6 & \quad 5 & \quad 5 & \quad 1 & \quad 7 & \quad 8 & \quad 9 & \quad 17
\end{align*}
\]
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is \textit{totally connected} in the standard base.

“Proof:” ($\Leftarrow$)

\[
\begin{array}{cccccccccc}
\times & 3 & \bullet & 9 & \bullet & 8 & \bullet & 6 & \bullet & 5 & \times & 5 & \times & 7 & \bullet & 1 & \times & 8 & \times & 9 & \times & 17
\end{array}
\]
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base \(\iff\) it is \textit{totally connected} in the standard base.

“Proof:” (\(\iff\))
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Arc diagram has only short arcs in some base $\iff$ it is totally connected in the standard base.

“Proof:” ($\Leftarrow$)
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\Leftarrow$)

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A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\iff$)
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base \(\iff\) it is *totally connected* in the standard base.

"Proof:" (\(\iff\))

\[
\begin{array}{cccccccccccc}
\times & 3 & \bullet & 9 & \bullet & 8 & \bullet & 6 & \times & 6 & \bullet & 6 & \times & 7 & \times & 8 & \times & 9 & \times & 17 & \bullet & a & \times & a & \leftrightarrow & \times & a + 1 & \bullet
\end{array}
\]
Example 1: Shortening

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"Proof:" ($\iff$)
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Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

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"Proof:" (⇐)

\[ a \quad a \quad \leftrightarrow \quad a + 1 \quad a + 1 \]

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A combinatorial model for highest weights of f.d. modules of \( \mathfrak{gl}(m, n) \)
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base \(\iff\) it is \textit{totally connected} in the standard base.

“Proof:” (\(\iff\))

\[ a \quad \iff \quad a + 1 \]

\[ \begin{array}{ccccccccccc}
3 & 9 & 8 & 6 & 6 & 7 & 8 & 9 & 17 & 6 & 1
\end{array} \]
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is *totally connected* in the standard base.

“Proof:” (⇐)

\[
\begin{array}{ccccccccccc}
\times & \bullet & \bullet & \times & \bullet & \times & \times & \times & \bullet & \bullet \\
3 & 9 & 8 & 7 & 7 & 8 & 9 & 17 & 6 & 1
\end{array}
\]
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\Leftarrow$)

\[ a \quad a \quad a + 1 \quad a + 1 \]

\[ \times \quad 3 \quad \bullet \quad \bullet \quad \times \quad 9 \quad 8 \quad 7 \quad 7 \quad 7 \quad \times \quad 8 \quad 9 \quad \times \quad 17 \quad 6 \quad 1 \]
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is *totally connected* in the standard base.

“Proof:” (⇐)

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\begin{array}{ccccccccc}
\times & \bullet & \times & \bullet & \times & \bullet & \times & \bullet & \times \\
3 & 9 & 7 & 8 & 7 & 7 & 8 & 9 & 17
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“Proof:” ($\iff$)

\[
\begin{array}{cccccccc}
\times & \times & \bullet & \bullet & \times & \times & \times & \bullet \\
3 & 7 & 9 & 8 & 7 & 7 & 8 & 9 & 17 & 6 & 1
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“Proof:” ($\iff$)

\[
\begin{array}{cccccccccc}
\times & \times & \bullet & \bullet & \times & \bullet & \times & \bullet & \times & \bullet \\
3 & 7 & 9 & 8 & 8 & 8 & 9 & 17 & 6 & 1
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Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\Leftarrow$)
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Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\iff$)

\[ a \quad \longleftrightarrow \quad a + 1 \]

\[ \begin{array}{ccccccccccccccc}
3 & \times & 7 & 9 & \times & 9 & \times & 9 & \times & 17 & 6 & 1 \\
\end{array} \]
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is totally connected in the standard base.

“Proof:” (⇐)

(⇒)
Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\Leftarrow$)

```
3 7 9 9 9 9 17 9 9 9 9 17 6 1
```

($\Rightarrow$)

```
a a a + 1 a + 1
```

or

```
a a a + 1 a + 2 a + 3 a + 3
```

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A combinatorial model for highest weights of f.d. modules of $\mathfrak{gl}(m, n)$
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base $\iff$ it is *totally connected* in the standard base.

“Proof:” ($\Leftarrow$)

Check that moves required to reach standard base preserve “interval property.”

(⇒)

\[\begin{array}{cccccc}
3 & 7 & 9 & 9 & 9 & 9 \\
\end{array}\]
Example 1: Shortening

Proposition 1 (C., Hoyt, Reif)

Arc diagram has only short arcs in some base ⇔ it is *totally connected* in the standard base.

“Proof:” (⇐)

(⇒)
Check that moves required to reach standard base preserve "interval property."
Example 2: Nests

Nest: Part of arc diagram under outermost arc.
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Nest: Part of arc diagram under outermost arc.

Proposition 2 (C., Hoyt, Reif)

If an arc diagram has at least two nests, then there exists a base where all arcs are short.
Example 3: Kac-Wakimoto character formula

Weyl character formula: For a semisimple Lie algebra,

\[ ch L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w(e^{\lambda + \rho}) \frac{e^\rho}{e^\rho R}. \]
Example 3: Kac-Wakimoto character formula

(Kac ‘77): If arc diagram has no arcs then

\[ \text{ch } L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w (e^{\lambda + \rho}) \cdot e^{\rho} R. \]
Example 3: Kac-Wakimoto character formula

(Bernstein, Leites ‘80): If arc diagram has one arc $\beta$ then

$$
ch L(\lambda) = \frac{\sum_{w \in W} (\text{sgn } w) \cdot w \left( \frac{e^{\lambda + \rho}}{1 + e^{-\beta}} \right)}{e^\rho R}.
$$
(Kac-Wakimoto ’94): Conjecturally, if arc diagram has only short arcs then

$$\text{ch } L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w \left( \frac{e^{\lambda + \rho}}{\prod_{\beta \in S} (1 + e^{-\beta})} \right) \left( |S| ! e^\rho R \right).$$
Theorem 1 (C., Hoyt, Reif)

If arc diagram has only short arcs then

\[ \text{ch } L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w \left( \frac{e^{\lambda + \rho}}{\prod_{\beta \in S} (1 + e^{-\beta})} \right) \frac{(|S|)!e^\rho R}{(|S|)!}. \]
Example 3: Kac-Wakimoto character formula

**Theorem 1 (C., Hoyt, Reif)**

If arc diagram has only short arcs then

\[
ch L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w \left( \frac{e^{\lambda+\rho}}{\prod_{\beta \in S} (1+e^{-\beta})} \right) \cdot \frac{(|S|)!}{e^\rho R}.
\]

“Proof:”
Example 3: Kac-Wakimoto character formula

Theorem 1 (C., Hoyt, Reif)

If arc diagram has only short arcs then

\[
\text{ch } L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w \left( \frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1+e^{-\beta})} \right) \cdot \frac{(|S|)!e^\rho R}{e^\rho R}. 
\]

Theorem 1 (C., Hoyt, Reif)

If arc diagram has only short arcs then

\[ \text{ch } L(\lambda) = \sum_{w \in W} (\text{sgn } w) \cdot w \left( \frac{e^{\lambda+\rho}}{\prod_{\beta \in S} (1 + e^{-\beta})} \right) \cdot (\prod_{\beta \in S} (1 + e^{-\beta})) \cdot (|S|)! \cdot e^\rho R. \]

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(C., Hoyt, Reif): See how SZ formula changes with each step of shortening algorithm. In the end, it is the Kac-Wakimoto formula.
Example 4: Determinantal character formula

Theorem 2 (C., Hoyt, Reif)

For a totally connected arc diagram:

\[
\begin{array}{cccccccc}
9 & 7 & 5 & 1 & 3 & 5 & 6 & 7 \\
\end{array}
\]

we have, with the convention \( x_i = e^{\varepsilon_i}, y_j = e^{-\delta_j} \):

\[
e^\rho R \text{ch} L(\lambda) = \pm \begin{vmatrix}
 x_1^9 x_1^1 f(x_1, y_1) & f(x_1, y_2) & f(x_1, y_3) & f(x_1, y_4) \\
 x_2^9 x_2^1 f(x_2, y_1) & f(x_2, y_2) & f(x_2, y_3) & f(x_2, y_4) \\
 x_3^9 x_3^1 f(x_3, y_1) & f(x_3, y_2) & f(x_3, y_3) & f(x_3, y_4) \\
 x_4^9 x_4^1 f(x_4, y_1) & f(x_4, y_2) & f(x_4, y_3) & f(x_4, y_4) \\
 0 & 0 & y_1^3 & y_2^3 & y_3^3 & y_4^3 \\
 0 & 0 & y_1^6 & y_2^6 & y_3^6 & y_4^6
\end{vmatrix},
\]

where \( f(x_i, y_j) = \frac{(x_i y_j)^7}{1 + (x_i y_j)^{-1}} \).
Further work

To do:

- Are there particularly nice bases for non-tc weights? (In the spirit of Proposition 2).
- Classify all tame modules (ones for which the Kac-Wakimoto formula is correct in some base).
- Obtain a character formula for any base.
- Generalize to other Lie types.
- Connect the determinantal formula to supersymmetric functions.
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