1. (7 Points) Find a general solution to the differential equation

$$(D^3 - 27I)y = 0$$

[Recall: $D$ is the operator that takes a derivative ($Dy = y'$), and $I$ is the identity operator ($Iy = y$).]

The characteristic equation is $r^3 - 27 = 0$. There should be 3 roots, and some of them will likely be complex. To find them, write

$$r^3 = 27$$

and then write $27$ in polar complex form as $27 = 3 \cdot e^{i\pi}$, where $n$ is an integer:

$$r = 3 \cdot e^{i\frac{2\pi}{3}}$$

Now let $n = 0$, $n = 1$, and $n = 2$:

$n = 0 \rightarrow r = 3 \cdot e^{i0} = 3$

$n = 1 \rightarrow r = 3 \cdot e^{i\frac{2\pi}{3}} = 3 \cos \frac{2\pi}{3} + 3i \sin \frac{2\pi}{3} = -\frac{3}{2} + \frac{3\sqrt{3}}{2} i$

$n = 2 \rightarrow r = 3 \cdot e^{i\frac{4\pi}{3}} = -\frac{3}{2} - \frac{3\sqrt{3}}{2} i$

So roots are $3, -\frac{3}{2} \pm \frac{3\sqrt{3}}{2} i$.

Basis functions for solution space are $e^{3x}, e^{\frac{3x}{2}} \cos \left(\frac{3\sqrt{3}}{2} x\right), e^{\frac{-3x}{2}} \sin \left(\frac{3\sqrt{3}}{2} x\right)$

General solution:

$$Y(x) = C_1 e^{3x} + C_2 e^{\frac{3x}{2}} \cos \left(\frac{3\sqrt{3}}{2} x\right) + C_3 e^{\frac{-3x}{2}} \sin \left(\frac{3\sqrt{3}}{2} x\right)$$
2. (3 Points) Consider the non-homogenous differential equation
\[ y'' - 4y = 12 \]

Given that \( y_h = c_1e^{2x} + c_2e^{-2x} \) solves the associated homogeneous equation \( y'' - 4y = 0 \) and given that \( y_p = -3 \) is a particular solution, write a general form for the solution to the DEQ above.

\[ Y(x) = y_h + y_p \] (The general solution is the general homogeneous solution plus a particular solution)

\[ Y(x) = c_1e^{2x} + c_2e^{-2x} - 3 \]

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Extra Credit

(Earn back up to 2 points) Write a correct sentence that uses the words “basis”, “dimension”, “span”, and “linearly independent”.

E.g. If \( V \) is a vector space of dimension \( n \), then any basis for \( V \) consists of \( n \) elements of \( V \) that both span \( V \) and are linearly independent.