Math 1271: Review for Midterm 2

The following problems are meant to provide challenging practice with concepts that are fair game for the midterm. They are by no means representative of everything that might be on the test.

3.3 Derivatives of Trigonometric Functions

Find \( \lim_{x \to 0} \frac{(\sin 2x)(\sin 4x)(\sin 6x)}{x^3} \)

48

(Spring 2008 Final Exam)

3.4 The Chain Rule - See Assorted Derivatives, below

Find \( y' = \frac{dy}{dx} \) where \( x^2 + y^2 = e^y \)

\[ \frac{e^y y' - 2x}{2y - e^y x} \]

(Spring 2008 Final Exam)

Find the equation of the tangent line to \( (x^2 + y^2 - 1)^3 - x^2 y^3 = 0 \) at the point \((1,0)\) .

This curve doesn’t have a tangent at \((1,0)\); see solutions

3.6 Derivatives of Logarithmic Functions

Write down the equation of the tangent line to the graph of \( f(x) = x^2 \) at \( x = 1 \).

\[ y = x \]

@Spring 2008 Final Exam

Assorted Derivatives

Find the following derivatives ((d) from Spring 2007 Final Exam):

\[
\begin{align*}
(a) \quad & \frac{d}{dt} \left( \ln \left( \frac{1}{t} \right) \cos t \right) \\
(b) \quad & \frac{d}{dy} \frac{17^y}{y^4} \\
(c) \quad & \frac{d}{dy} \left( 5 \arccos(y^2) + \cos^2 3y \right) \\
(d) \quad & \frac{d}{dx} \left( \arctan(2x)^2 \right)
\end{align*}
\]

3.9 Related Rates

To let off steam after their midterm, Alice and Bob run out the door of their stuffy building. Alice runs straight North, while Bob runs straight East. Alice covers 2 m/s and Bob covers 1.5 m/s. How fast is the distance between them increasing 1 minute after they exit the building?

\[ \frac{5}{2} \text{ m/s} \]

3.10 Linear Approximations

Estimate \( \frac{1}{\sqrt{(1.99)^2 + 5}} \) by the method of linear approximation.

\[ \approx 0.334 \]

(Spring 2007 Final Exam)

4.1 Maximum and Minimum Values

For the function \( g(x) = -x^2 - 4x + 5 \) on the interval \([-3,0]\), find:

(a) the local minima and maxima \( \text{Local max at } (-2,9), \text{ no local min.} \)

(b) the global (absolute) minima and maxima \( f(0) = 5 \), \( f(-2) = 9 \)
4.2 The Mean Value Theorem

You are given a function \( f(x) \) that is continuous and differentiable on the interval \([-1, 2]\). You are also told that \( f(-1) = -5 \) and \( f(2) = 7 \). Which of the following statements about \( f(x) \) is NOT necessarily true?

(A) \( f(x) \) has an absolute max for some \( x \) on \([-1, 2]\)
(B) There is a point \( c \) with \(-1 < c < 2\) where \( f'(c) = 0 \).
(C) There is a point \( c \) with \(-1 < c < 2\) where \( f'(c) = 4 \).
(D) \( f'(x) > 0 \) for all values of \( x \) between \(-1\) and \( 2 \).
(E) Any of the statements A, B, C, or D can be false depending on what properties \( f(x) \) has.

(modified from Spring 2008 Final Exam)

4.3 How Derivatives Affect the Shape of a Graph

Examine the function \( f(x) = xe^{-2x^2} \).

(i) Find where \( f(x) \) is increasing and decreasing and has local max or min.
(ii) Find where \( f(x) \) is concave up and down and has points of inflection.
(iii) Find \( \lim_{x \to \infty} f(x) \) as \( x \to \infty \) and \( x \to -\infty \).

4.4 Indeterminate Forms and L'Hospital's Rule

Find \( \lim_{x \to 0} \frac{\sin(\sin(x))}{x} \) 1

Find \( \lim_{x \to 1} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right) \) \( \frac{1}{2} \)

(Spring 2008 Final Exam)
MATH 1271: REVIEW FOR MIDTERM 2

SOLUTION GUIDE

\[
\lim_{x \to 0} \frac{(\sin 2x)(\sin 4x)(\sin 6x)}{x^3} = \lim_{x \to 0} \frac{\sin 2x \cdot \sin 4x \cdot \sin 6x}{x^3}
\]

\[
= \lim_{x \to 0} \frac{\sin 2x}{x^3} \cdot \lim_{x \to 0} \frac{\sin 4x}{x} \cdot \lim_{x \to 0} \frac{\sin 6x}{x^3}
\]

\[
= 2 \left( \lim_{x \to 0} \frac{\sin 2x}{2x} \right) \cdot 4 \left( \lim_{x \to 0} \frac{\sin 4x}{4x} \right) \cdot 6 \left( \lim_{x \to 0} \frac{\sin 6x}{6x} \right) = 2 \cdot 4 \cdot 1 \cdot 6 \cdot 1 = 48
\]

RECALL: \( \lim_{y \to 0} \frac{\sin y}{y} = 1 \) (Here let \( y = 2x, 4x, \) or \( 6x \))

Prob #1

\[
X^2 + y^2 = e^{xy}
\]

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(e^{xy})
\]

\[
2x + 2y \frac{dy}{dx} = e^{xy} \left( y + x \frac{dy}{dx} \right)
\]

Chain Rule

\[
2x + 2y \frac{dy}{dx} = e^{xy} y + e^{xy} x \frac{dy}{dx}
\]

\[
2y \frac{dy}{dx} - e^{xy} x \frac{dy}{dx} = e^{xy} y - 2x
\]

\[
\frac{dy}{dx} \left( 2y - e^{xy} x \right) = e^{xy} y - 2x
\]

\[
\frac{dy}{dx} = \frac{e^{xy} y - 2x}{2y - e^{xy} x}
\]

Prob #2

Want equation of tangent line to \((x^2 + y^2 - 1)^3 - x^2 y^2 = 0\) at \((1,0)\).

Need: slope of tangent line (= \(\frac{dy}{dx}\)) and point on tan. line (=\((1,0)\))

To find \(\frac{dy}{dx}\), use implicit differentiation:

\[
\frac{d}{dx} \left[ (x^2 + y^2 - 1)^3 - x^2 y^2 \right] = \frac{d}{dx} [0]
\]

\[
3(x^2 + y^2 - 1)^2 \cdot (2x + 2y \frac{dy}{dx}) - (2x y^3 + x^2 3y^2 \frac{dy}{dx}) = 0
\]

Chain Rule

\[
(3(x^2 + y^2 - 1)^2 \cdot 2x) + 3(x^2 + y^2 - 1)^2 \cdot 2y \frac{dy}{dx} - 2xy^3 - x^3 3y^2 \frac{dy}{dx} = 0
\]

\[
6x (x^2 + y^2 - 1)^2 + 6y (x^2 + y^2 - 1) \frac{dy}{dx} - 2xy^3 - 3x^3 y^2 \frac{dy}{dx} = 0
\]

\[
(6y (x^2 + y^2 - 1)^2 - 3x^3 y^2) \frac{dy}{dx} = 2xy^3 - 6x (x^2 + y^2 - 1)^2
\]

\[
\frac{dy}{dx} = \frac{2xy^3 - 6x (x^2 + y^2 - 1)^2}{6y (x^2 + y^2 - 1)^2 - 3x^3 y^2}
\]
we want \( \frac{dy}{dx} \) at the point \( x=1, \ y=0 \), so plug these values in:

\[
\frac{dy}{dx} = \frac{2 \cdot 1 \cdot 0^3 - 6 \cdot 1 (1^2 + 0^2 - 1)^2}{6 \cdot 0 (1^2 + 0^2 - 1)^2 - 3 \cdot 1^3 \cdot 0^2} = 0 - 0 = 0
\]

This problem did not turn out the way I expected.

The relation \( (x^2 + y^2 + 1)^3 - x^2 y^3 = 0 \) has a heart-shaped graph and I expected a tangent line to exist at \((1,0)\).

However, if you zoom in on \((1,0)\), you see a small pointy part there. Apparently no derivative exists at \((1,0)\), so no tangent line exists. If I had written it this way intentionally, this would be a "trick problem".

In general, if you can find the derivative at a point \((x_0, y_0)\), say \( \frac{dy}{dx} = m \) at \((x_0, y_0)\), then the equation for the tangent line at \((x_0, y_0)\) is (in point-slope form):

\[
y - y_0 = m \ (x - x_0)
\]

3.6 Write the equation of the tangent line to the graph of \( f(x) = x^{x^2} \) at \( x=1 \).

As before, we need the slope of the tangent line (i.e. the derivative at \( x=1 \)) and the point \((1, f(1)) = (1, 1^2) = (1, 1)\)

Let \( y = x^{x^2} \). To find \( \frac{dy}{dx} \), we use logarithmic differentiation because \( x \) appears in the base and the exponent of the expression.

\[
\ln(y) = \ln(x^{x^2}) \quad \text{Take \( \ln \) both sides}
\]

\[
\ln(y) = x^2 \ln(x) \quad \text{use \( \ln \) rules: \( \ln(a^b) = b \ln(a) \)}
\]

\[
\frac{1}{y} \frac{dy}{dx} = 2x \ln(x) + x^2 \cdot \frac{1}{x} \quad \text{Differentiate w.r.t. \( x \) on both sides}
\]

\[
\frac{dy}{dx} = y \left[ 2x \ln(x) + x^2 \cdot \frac{1}{x} \right] = x^{x^2} \left[ 2x \ln(x) + x \right]
\]
So at \( x = 1 \), \( \frac{dy}{dx} = 1^2 [2 \cdot 1 \cdot \ln(1) + 1] = 1 \), so the slope of the tangent line at \((0,1)\) is \( m = 1 \). 

And plugging in to the point-slope formula for a line, we get \( (y - 1) = 1 \cdot (x - 1) \) as the equation of the tangent (equivalently, \( y = x \)).

### Assorted Derivatives

(a) \( \frac{d}{dt} \left( \ln \left( \frac{1}{e} \right)^\cos t \right) \)

(assume parentheses where I’ve drawn them; sorry this was ambiguous)

Let \( y = \left( \ln \left( \frac{1}{e} \right) \right)^\cos t \). We want \( \frac{dy}{dt} \) and we’ll need logarithmic differentiation.

\[
\begin{align*}
\ln(y) &= \ln \left( \left( \ln \left( \frac{1}{e} \right) \right)^\cos t \right) \\
\ln(y) &= \cos t \ln \left( \ln \left( \frac{1}{e} \right) \right) \\
\frac{1}{y} \frac{dy}{dt} &= (-\sin t) \left( \ln \left( \ln \left( \frac{1}{e} \right) \right) \right) + \left( \cos t \right) \frac{1}{\ln \left( \frac{1}{e} \right) \cdot \frac{1}{1/e} \cdot \frac{-1}{e}} \\
\frac{dy}{dt} &= \frac{y \cdot \left( -\sin t \ln(\ln(\frac{1}{e})) + \frac{-\cos t}{\frac{1}{e} \ln(\frac{1}{e})} \right)}{\left( \ln \left( \frac{1}{e} \right) \right)^\cos t} \\
\end{align*}
\]

Plug in for \( y \):

\( \frac{dy}{dt} = \left( \ln \left( \frac{1}{e} \right) \right)^\cos t \cdot \left[ -\sin t \ln(\ln(\frac{1}{e})) + \frac{-\cos t}{\frac{1}{e} \ln(\frac{1}{e})} \right] \)

(b) \( \frac{d}{d\theta} \left[ \frac{17^\theta}{17^\theta \ln(17)} \right] \)

Don’t let the \( \theta \) unnerve you! It’s a variable just like \( x \); you could replace it w/ \( x \).

Quotient Rule: \( \frac{17^\theta \ln(17) - 17^\theta (17^\theta \ln(17))}{(17^\theta)^2} \) if you like.
\( \frac{d}{dy} \left[ 5 \arccos(y^2) + \cos^2(3y) \right] \)

\[ = \frac{d}{dy}[5 \arccos(y^2)] + \frac{d}{dy}[\cos^2(3y)] \]

\[ = 5 \cdot \frac{-1}{\sqrt{1-(y^2)^2}} \cdot (2y) + 2(\cos(3y))(-\sin(3y)) \cdot 3 \]

Chain Rule
Power Rule
Since \( \cos^2(3y) = (\cos 3y)^2 \)

\[ = \frac{-10y}{\sqrt{1-y^4}} - 6 \cos(3y) \sin(3y) \]

\( \frac{d}{dx} \left[ \left( \arctan(2x) \right)^2 \right] = 2 \left( \arctan(2x) \right) \cdot \frac{1}{1+(2x)^2} \cdot 2 \)

Chain Rule on outermost function
Power Rule on \( \arctan \)

\[ = \frac{4 \arctan(2x)}{1+4x^2} \]

**3.9**

Let \( a \) be the distance Alice has run in \( t \) seconds after leaving the building.
Let \( b \) be the distance Bob has run in the same \( t \) seconds.

Have: \( a(t) = 2t \) (distance = rate \( \times \) time)
\( b(t) = 1.5t \)

Let \( c \) be the distance between Alice and Bob.
Pythagorean Thm gives: \( C^2 = a^2 + b^2 \)
\( \text{or } C = \sqrt{a^2 + b^2} \)
Differentiating both sides of \( C = \sqrt{a^2 + b^2} \) with respect to \( t \) (and remembering \( a, b, \) and \( c \) are not constants but function) of \( t \), we have

\[
\frac{dc}{dt} = \frac{1}{2} (a^2 + b^2)^{-1/2} \left( 2a \frac{da}{dt} + 2b \frac{db}{dt} \right)
\]

At \( t = 60 \) seconds, \( a = 2 \cdot 60 = 120 \) and \( b = 1.5 \cdot 60 = 90 \).
We also can find \( \frac{da}{dt} = 2 \) and \( \frac{db}{dt} = 1.5 \).

So at \( t = 60 \) sec,

\[
\frac{dc}{dt} = \frac{1}{2} \left( (120)^2 + (90)^2 \right)^{-1/2} \left( 2 \cdot 120 \cdot 2 + 2 \cdot 90 \cdot 1.5 \right)
\]

\[
= \frac{5}{2} \text{ m/s}
\]

3.10 Estimate \( \sqrt{(1.99)^2 + 5} \) by linear approximation.

Let's consider the function \( f(x) = \sqrt{x^2 + 5} \).

We want to estimate \( f(1.99) \).

Notice 1.99 is very close to 2, and \( f(2) = \sqrt{2^2 + 5} = \sqrt{1} = \frac{1}{3} \).

Let's find the tangent line to \( f(x) \) at \( (2, \frac{1}{3}) \) and find the y-value on the tangent line at \( x = 1.99 \). Since the tangent line is close to the function near \( x = 2 \), this will be a good approximation.

For tangent line, need \( f'(x) = -\frac{1}{2} (x^2 + 5)^{-3/2} \cdot 2x \)

so \( f'(2) = -\frac{1}{2} (2^2 + 5)^{-3/2} \cdot (2 \cdot 2) \)

\[= -\frac{1}{2} \cdot \frac{1}{3} \cdot 4 \cdot y^2 = -\frac{2}{27} \]

Equation of tangent : \( (y - \frac{1}{3}) = \frac{-2}{27} (x - 2) \) \( \rightarrow y = \frac{1}{3} - \frac{2}{27} (x - 2) \)
at $x = 1.99$, 
\[ y = \frac{1}{3} - \frac{2}{27} (1.99 - 2) \]
\[ = \frac{1}{3} - \frac{2}{27} (-0.01) \]
\[ = \frac{1}{3} + \frac{0.02}{27} \]
\[ \approx 0.334 \]

4.1 $g(x) = -x^2 - 4x + 5$ on $[-3, 0]$

(a) local min/max can occur where $g'(x)$ DNE (won't happen for this polynomial, which is differentiable everywhere) or where $g'(x)$ switches from positive to negative.

\[ g'(x) = -2x - 4 \quad \text{so} \quad g'(x) = 0 \text{ at } x = -2. \]

\[ g'(x) \begin{array}{c}
\leftarrow & -2 \\
(0) & 0 \\
\rightarrow & 1
\end{array} \]

local max occurs at $(-2, g(-2)) = (-2, 9)$

no local min.

(b) global max/min can occur where $g'(x) = 0$, $g'(x)$ DNE, and end points. From (a) we know $g'(x) = 0$ at $x = -2$ and $g'(x)$ DNE nowhere. Endpoints are -3 and 0.

check values of $f$ at candidate points:

\[ f(-2) = 9 \quad \text{absolute max} \]
\[ f(-3) = 8 \]
\[ f(0) = 5 \quad \text{absolute min} \]

4.2 (D)

Note (A) is true (this is a theorem from 4.1)
(B) is true due to the Intermediate Value Thm
(C) is true due to the Mean Value Thm
4.3 \[ f(x) = xe^{-2x^2} \]

(i) \[ f'(x) = e^{-2x^2} + xe^{-2x^2} (-4x) \]
\[ = e^{-2x^2} (1 - 4x^2) \]
\[ f'(x) = 0 \quad \text{where} \quad 1 - 4x^2 = 0 \quad (e^{-2x} \text{ never 0}) \]
\[ 1 = 4x^2 \]
\[ x^2 = \frac{1}{4} \]
\[ x = \pm \frac{1}{2} \]

\[ f'(x): \bigcirc \quad \bigcirc \quad \bigcirc \]

\[ f \text{ is increasing on } (-\frac{1}{2}, \frac{1}{2}) \text{ and} \]
\[ f \text{ is decreasing on } (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty) \]
\[ f \text{ has a local min at } (-\frac{1}{2}, f(-\frac{1}{2})) \]
\[ f \text{ has a local max at } (\frac{1}{2}, f(\frac{1}{2})) \]

(ii) \[ f''(x) = \frac{d}{dx} \left( e^{-2x^2} (1 - 4x^2) \right) \]
\[ = e^{-2x^2} (-2)(1 - 4x^2) + e^{-2x^2} (-8x) \]
\[ = e^{-2x^2} (-2 + 8x^2 - 8x) \]
\[ = e^{-2x^2} (8x^2 - 8x - 2) \]
\[ = 2e^{-2x^2} (4x^2 - 4x - 1) \]

Conditions
\[ f''(x) = 0 \quad \text{where} \quad 4x^2 - 4x - 1 = 0 \]

Quadratic formula:
\[ x = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1}{2} \pm \frac{1}{2} \sqrt{2} \]
\[ x = \frac{1+\sqrt{2}}{2} \text{ or } x = \frac{1-\sqrt{2}}{2} \]
\[ f''(x) \]

\[ \frac{1 - \frac{1}{2}}{2} \quad \frac{1 + \frac{1}{2}}{2} \]

- \( F \) is concave up on \((-\infty, \frac{1 - \frac{1}{2}}{2}) \cup (\frac{1 + \frac{1}{2}}{2}, \infty)\)
- \( F \) is concave down on \((\frac{1 - \frac{1}{2}}{2}, \frac{1 + \frac{1}{2}}{2})\)
- \( F \) has inflection points at \( \frac{x}{2} = \frac{1 - \frac{1}{2}}{2} \) and \( \frac{x}{2} = \frac{1 + \frac{1}{2}}{2} \).

\[
\begin{align*}
(iii) \lim_{x \to \infty} x e^{-2x^2} & \quad \text{going to } \infty \\
& \quad \text{going to } -\infty \\
& \quad \text{going to } 0
\end{align*}
\]

"\( \infty \cdot 0 \)" is indeterminate form.

Try to massage into form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) and use L'Hôpital.

\[
\begin{align*}
\lim_{x \to \infty} x e^{-2x^2} & \to \infty \\
\lim_{x \to \infty} \frac{x}{e^{2x^2}} & \to 0 \\
\lim_{x \to \infty} \frac{1}{e^{2x^2} (4x)} & \to -\infty
\end{align*}
\]

Here we used L'Hôpital: take deriv. of numerator and denominator.

\[
\begin{align*}
\lim_{x \to \infty} x e^{-2x^2} & \quad \text{still going to } 0 \\
& \quad \text{going to } -\infty
\end{align*}
\]

"\(-\infty \cdot 0\)" is also indeterminate.

Proceed similarly to above.

\[
\begin{align*}
\lim_{x \to \infty} \frac{x}{e^{2x^2}} & \to -\infty \\
\lim_{x \to \infty} \frac{1}{e^{2x^2} (4x)} & \to -\infty
\end{align*}
\]

\[
= 0
\]
4.4

Prob #1
\[
\lim_{t \to 0} \frac{\sin (\sin (\sin t))}{t} = 0 \quad \text{use L'Hospital}
\]
\[
= \lim_{t \to 0} \cos (\sin (\sin t)) \cdot \cos (\sin (\sin t)) \cdot \cos t \quad \text{\( \leftarrow \) derivative of numerator}
\]
\[
= \lim_{t \to 0} \frac{\cos (\sin (\sin t)) \cdot \cos (\sin (\sin t)) \cdot \cos (\sin (\sin t))}{1} \quad \text{\( \leftarrow \) deriv. of denominator}
\]
\[
= \frac{\cos (\sin (\sin (0))) \cdot \cos (\sin (\sin (0))) \cdot \cos (\sin (\sin (0)))}{1} \quad \text{Since trig. fn's cos, sin are continuous}
\]
\[
= 1
\]

Prob #2
\[
\lim_{x \to 1} \frac{1 - \frac{1}{x}}{ln x - x - 1} \quad \text{looks like } \frac{\frac{1}{0} - \frac{1}{0}}{0 - 0} = \frac{\infty}{\infty}
\]
\text{an indeterminate form. Get a common denominator, massage into form } \frac{\infty}{\infty} \quad \text{or } \frac{0}{0}
\text{then use L'Hospital.}
\[
= \lim_{x \to 1} \frac{x - 1 - \ln x}{(\ln x)(x - 1)} = 0
\]
\[
H = \lim_{x \to 1} \frac{1 - \frac{1}{x}}{(\ln x)(x - 1) + \ln x(1)}
\]
\[
= \lim_{x \to 1} \frac{1/x^2}{1/x^2 + 1/x}
\]
\[
= \frac{1}{1 + 1} = \frac{1}{2}
\]