Let $G = (V, E)$ be graph. Recall that a $k$-clique is a set of $k$ vertices of $G$ each pair of which is connected by an edge. A $k$-independent set is a set of $k$ vertices of $G$ no pair of which is connected by an edge. Ramsey theory says that a really big graph (i.e. one with a whole lot of vertices) must have either a moderately big clique or a moderately big independent set. The following is a first result along these lines.

**Proposition 1.** Every graph with 6 vertices contains either a 3-clique or a 3-independent set.

**Proof.** Let $G = (V, E)$ with $|V| = 6$ and pick any vertex $v \in V$. There are 5 other vertices, so by pigeonhole, either $v$ has 3 neighbors or 3 non-neighbors. First suppose $v$ has 3 neighbors, say $v_1, v_2, v_3$. If there is some edge $v_i v_j \in E$, then $\{v, v_i, v_j\}$ is a 3-clique. If no such edge exists then $\{v_1, v_2, v_3\}$ form a 3-independent set.

Now suppose $v$ has 3 non-neighbors, say $v_1, v_2, v_3$. If all 3 possible edges among $v_1, v_2, v_3$ are present, then $\{v_1, v_2, v_3\}$ form a 3-clique. Otherwise there is some missing edge $v_i v_j \notin E$. In this case $\{v, v_i, v_j\}$ is a 3-independent set. \[\square\]

Note that the graph $C_5$ has neither a 3-clique nor a 3-independent set:

Combined with the previous Proposition, this implies that 6 is the minimum number of vertices needed in a graph to ensure that the graph has either a 3-clique or a 3-independent set. This is all a bit of a mouthful, so we will shorten it to $r(3, 3) = 6$. In general, $r(a, b) = n$ means “$n$ is the minimum number of vertices needed in a graph to ensure that the graph has either an $a$-clique or a $b$-independent set. The numbers $r(a, b)$ are called the Ramsey numbers and are some of the most mysterious positive integers in combinatorics.

**Exercise 1.**

- $r(a, 1) = r(1, b) = 1$ for all $a, b \geq 1$
- $r(a, 2) = a, r(2, b) = b$ for all $a, b \geq 1$
- $r(a, b) = r(b, a)$ for all $a, b \geq 1$

It is not immediately clear from the definition that $r(a, b)$ actually exists for each $a$ and $b$. This can be proven by induction on $a$ and $b$ with the following serving as the inductive case.

**Proposition 2.** Suppose $r(a - 1, b) = m$ and $r(a, b - 1) = n$. Then every graph with $m + n$ vertices has either an $a$-clique or a $b$-independent set. Therefore $r(a, b)$ exists
and in fact
\[
   r(a, b) \leq r(a - 1, b) + r(a, b - 1)
\]

Proof. Let \( G = (V, E) \) with \(|V| = m + n\) and pick any \( v \in V \). There are \( m + n - 1 \) other vertices, so \( v \) has either \( m \) neighbors or \( n \) non-neighbors. Suppose \( v \) has \( m \) neighbors, \( v_1, \ldots, v_m \). Let \( G' \) be the subgraph of \( G \) with vertices \( v_1, \ldots, v_m \) together with all edges of \( G \) between them. Since \( r(a-1, b) = m \), either \( G' \) has an \((a-1)\)-clique or a \( b \)-independent set. If \( G' \) has an \((a-1)\)-clique then adding \( v \) (which is adjacent to everything in \( G' \)) makes an \( a \)-clique of \( G \). Otherwise \( G' \) has a \( b \)-independent set which is itself a \( b \)-independent set of \( G \) as well.

The case where \( v \) has \( n \) non-neighbors is similar. \( \square \)

We now know some Ramsey numbers, along with a way to bound any Ramsey numbers in terms of its predecessors. The following table sums up what we can conclude about \( r(m, n) \) for \( m, n \leq 5 \)

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For example, \( r(3, 4) \leq r(2, 4) + r(3, 3) = 4 + 6 = 10 \). This table is built up the same way as Pascal’s triangle, so the entries are all binomial coefficients.

**Theorem 1.** \( r(m, n) \leq \binom{m+n-2}{m-1} \) for all \( m, n \geq 1 \).

In all examples we have seen, equality holds. It begins to break immediately thereafter, for instance \( r(3, 4) = 9 \) (compare to 10 in the table) and \( r(4, 4) = 18 \) (compare to 20 in the table). However, finding better bounds is considered an extremely difficult problem. Even working out individual Ramsey numbers is hard. For instance, the value of \( r(5, 5) \) is currently unknown.