Chapter 3

Problem 4. Partition the set $\{1, 2, \ldots, 2n\}$ down into $n$ subsets
$$\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}.$$ Because $n + 1$ numbers are chosen from among these $n$ sets, by the pigeonhole principle some set must have both of its elements chosen. These two elements will automatically differ by 1.

Problem 12. The Chinese remainder theorem states that if $m, n$ are relatively prime positive integers and $a, b$ are integers then there is always a solution $x$ to the system
$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}$$
There are many possible counterexamples if $m$ and $n$ are not assumed to be relatively prime. An extreme case would be to choose $m = n$. For example, if $m = n = 2$, $a = 0$, and $b = 1$ then the system looks like
$$x \equiv 0 \pmod{2}$$
$$x \equiv 1 \pmod{2}$$
Clearly there is no solution since $x$ cannot be even and odd at the same time.

Problem 16. Each person can be acquainted to 0, 1, 2, \ldots up to $n - 1$ people. This is a total of $n$ possibilities. If everyone is acquainted with at least one other, then 0 is ruled out so there are $n - 1$ possibilities left. By the pigeonhole principle some two of the $n$ people would need to know the same number of people. As such, assume from here on out there is someone who knows no one.

Similarly, if everyone had at least one person whom they did not know, then no one would know $n - 1$ people. There would again just be $n - 1$ possibilities and some pair would have to know the same number of people. As such, we can assume that there is someone who knows everyone else.

At this point we are assuming the existence of a person X who knows no one and a person Y who knows everyone. In particular Y knows X but X does not know Y. This violates an implicit assumption of the problem that acquaintance is symmetric. So we have a contradiction, and one of the two previously handled cases must occur.

Problem 18. Divide the square into four smaller squares each of side lengths one. These four squares cover the big one, but they are not disjoint. Indeed some pairs of squares intersect in a side and all of them include the point in the middle of the big square. There are plenty of ways to refine this to get a partition of the big square. For example, number the squares from 1 to 4 and assign each point to a set $A_i$ where $i$ is the smallest number of a square that the point belongs to. Then
$A_1, \ldots, A_4$ are a partition of the big square and $A_i$ is contained in little square $i$ for all $i$.

By the pigeonhole principle, if 5 points are selected then some two of them belong to the same set $A_i$. In particular, they are both contained in square $i$. But square $i$ has side length 1 so the farthest apart the points can be is $\sqrt{2}$ (which occurs if they are at opposite corners).

**Problem 20.** We want to show $K_{17} \rightarrow K_3, K_3, K_3$. In words, every three coloring of $K_{17}$ contains either a red triangle, a blue triangle, or a green triangle. Indeed, choose an arbitrary vertex $u$ of $K_{17}$. It is connected by an edge to each of the other vertices, so there are a total of 16 edges coming out of $u$. Each is given one of the three colors. By the pigeonhole principle, there are some 6 edges coming out of $u$ that have the same color. Assume without loss of generality that these edges are all green. Let $v_1, v_2, \ldots, v_6$ be the other endpoints of these 6 edges.

Suppose first that there exist $1 \leq i < j \leq 6$ such that the edge connecting $v_i$ to $v_j$ is green. Then $u, v_i, v_j$ are the vertices of a triangle with all edges green. The other possibility is that all edges connecting the $v_i$ are either red or blue. The vertices $v_1, \ldots, v_6$ together with the edges connecting them give a copy of $K_6$ inside the big graph. Assuming only red and blue are used, there must be either a red triangle or a blue triangle in this $K_6$. This is simply a statement of the fact $K_6 \rightarrow K_3, K_3$ which we have proven.

To sum up, if some $v_i, v_j$ are connected by a green edge then there is a green triangle, and otherwise there is either a red or blue triangle.

**Chapter 4**

**Problem 5.** The number $k = b_1 + b_2 + \ldots + b_n$ gives the total number of inversions in the permutation, i.e., the total number of pairs of integers for which the larger appears first. Switching two adjacent numbers in the permutation changes the order of those two, but preserves the order of every other pair. If the adjacent numbers were originally in increasing order, they are now in decreasing order, and vice versa. The effect on $k$ is to increase it or decrease it by exactly 1.

If $k = 0$, then it is certainly true that you cannot reach $12 \cdots n$ in fewer than $k$ moves. Now suppose that $k > 0$ and that every permutation with $k - 1$ total inversions requires at least $k - 1$ moves to be brought to $12 \cdots n$. Let $i_1i_2 \cdots i_n$ be a permutation with $k$ total inversions. Suppose a sequence of $M$ moves can be found to bring $i_1i_2 \cdots i_n$ to $12 \cdots n$. We want to show that $M \geq k$. The total number of inversions starts at $k$, ends at 0 (since $12 \cdots n$ has no inversions) and changes by 1 each step. Therefore it must be equal to $k - 1$ at some point. It took at least 1 step to get to that point, and by the induction hypothesis, it takes at least $k - 1$ steps to get from there to $12 \cdots n$. Therefore it took at least $1 + (k - 1) = k$ steps in total.

**Online**

**Problem A.** (i) Suppose otherwise. Then each pair of the $n + 1$ people know each other and are hence on the same team. It follows that all $n + 1$ of the people are on the same team. This contradicts the fact that there are only $n$ people per team.
(ii) We are given $m+1$ people, each on one of the $m$ teams. By the pigeonhole principle, some two of them must be on the same team. Therefore these two people know each other.

(iii) The appropriate statement is $r(n+1, m+1) > mn$. Consider the graph $K_{mn}$ where the vertices represent the people from before. Color each edge red if the two people it connects know each other and green otherwise. Part (i) says that of any $n+1$ people, some two of them are unacquainted. In terms of the graph, this means that every $K_{n+1}$ has a green edge. Put another way, there is no red $K_{n+1}$. Similarly, part (ii) says that every $K_{m+1}$ in the graph contains a red edge so there is no green $K_{m+1}$.

The statement $r(n+1, m+1) = mn$ would mean that every coloring of $K_{mn}$ contains either a red $K_{n+1}$ or a green $K_{m+1}$. The graph described above provides a counterexample, so we can conclude $r(n+1, m+1) \neq mn$. In fact, it follows that more than $mn$ vertices are needed to ensure the desired property, so $r(n+1, m+1) > mn$. 