Chapter 4

Problem 19. One possibility is

000
001
011
010
110
100
101
111

All 8 length 3 bit strings are used exactly once, and each pair of consecutive strings differ in only 1 place. Therefore this is a Gray code. However, the last string 111 and the first string 000 differ in more than one place, so the code is not cyclic.

Problem 37. Let $x \in X$ be given. Then $xR'x$ and $xR''x$ by reflexivity. Therefore $xRx$. Hence $R$ is reflexive.

Let $x, y \in X$ and suppose that $xRy$ and $yRx$. Then in particular $xR'y$ and $yR'x$ so $x = y$, by antisymmetry of $R'$. Hence, $R$ is antisymmetric.

Let $x, y, z \in X$ and suppose $xRy$ and $yRz$. Then $xR'y$ and $yR'z$, so $xR'z$ by transitivity of $R'$. Similarly, $xR''y$ and $yR''z$, so $xR''z$ by transitivity of $R''$. It follows that $xRz$. Hence $R$ is transitive.

Chapter 5

Problem 10. Let $A$ be the set of pairs $(i, S)$ where $i \in S \subseteq \{1, 2, \ldots, n\}$ and $|S| = k$. There are $\binom{n}{k}$ possible choices for $S$, and any such choice of $S$ has exactly $k$ elements so there are always $k$ choices for $i$. By the multiplication principle

$$|A| = \binom{n}{k} k.$$

On the other hand, we can count $A$ another way by first selecting $i$. We know $i \in \{1, 2, \ldots, n\}$ so there are $n$ such choices. Fixing such a choice, we know that we must have $i \in S$ so it remains to decide on the $k-1$ elements of $S \setminus i$. These can be picked arbitrarily from $\{1, 2, \ldots, n\}$ except that we cannot take $i$ since it is already being used. So $S \setminus i$ can be any $k-1$ subset of the $n-1$ element set $\{1, 2, \ldots, n\} \setminus \{i\}$. The number of such choices is then $\binom{n-1}{k-1}$. By the multiplication principle,

$$|A| = n \binom{n-1}{k-1}.$$
Combined with the previous this implies
\[ k \binom{n}{k} = n \binom{n-1}{k-1} \]

**Problem 11.** Let \( S \) be an \( n \)-element set, and let \( a, b, c \in S \) be three distinct elements. There are \( \binom{n}{k} \) \( k \)-element subsets of \( S \) and \( \binom{n-3}{k} \) \( k \)-element subsets of \( S \setminus \{a, b, c\} \). Hence there are
\[ \binom{n}{k} - \binom{n-3}{k} \]
k-element sets that are contained in \( S \) but not in \( S \setminus \{a, b, c\} \). In other words, the left hand side of the problem represents the number of \( k \)-element subsets of \( S \) that include at least one of \( a, b, \) or \( c \).

Break such subsets into the following three cases:
1. subsets that contain \( a \)
2. subsets that do not contain \( a \) but do contain \( b \)
3. subsets that contain neither \( a \) nor \( b \), but do contain \( c \).

These cases are clearly mutually exclusive. The only case not covered is a subset that contains none of \( a, b, \) or \( c \), which are precisely the ones we wanted to avoid. It remains to count the number of subsets occurring in each case:

1. If a \( k \)-element subset contains \( a \), there are \( \binom{n-1}{k-1} \) choices for the remaining \( k-1 \) elements.
2. If a \( k \)-element subset contain \( b \) but not \( a \), then the other \( k-1 \) elements must come from the \( (n-2) \)-element set \( S \setminus \{a, b\} \). There are \( \binom{n-2}{k-1} \) possibilities.
3. If a \( k \)-element subset contains \( c \) but neither \( a \) nor \( b \), then the other \( k-1 \) elements must come from the \( (n-3) \)-element set \( S \setminus \{a, b, c\} \). There are \( \binom{n-3}{k-1} \) possibilities.

We have
\[ \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} \]
total subsets, which must equal \( \binom{n}{k} - \binom{n-3}{k} \) since they count the same thing.

**Problem 15.** We know that \( k \binom{n}{k} = n \binom{n-1}{k-1} \). Therefore
\[
\sum_{k=1}^{n} (-1)^{k-1} k \binom{n}{k} = \sum_{k=1}^{n} (-1)^{k-1} n \binom{n-1}{k-1} \\
= n \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \\
= n \sum_{j=0}^{n-1} (-1)^{j} \binom{n-1}{j}
\]
where we have made the subsitution \( j = k - 1 \). We have proven
\[
\sum_{j=0}^{n-1} (-1)^{j} \binom{n-1}{j} = 0
\]
so the original sum is also 0.

**Problem 19.** First

\[
2 \binom{m}{2} + \binom{m}{1} = \frac{2m(m - 1)}{2} + m = m(m - 1) + m = m^2.
\]

Therefore

\[
\sum_{m=1}^{n} m^2 = \sum_{m=1}^{n} \left( 2 \binom{m}{2} + \binom{m}{1} \right)
\]

\[
= 2 \sum_{m=1}^{n} \binom{m}{2} + \sum_{m=1}^{n} \binom{m}{1}
\]

\[
= 2 \sum_{m=2}^{n} \binom{m}{2} + \sum_{m=1}^{n} \binom{m}{1}
\]

where the last modification is justified because \(\binom{1}{2} = 0\). We can apply the identity

\[
\sum_{m=k}^{n} \binom{m}{k} = \binom{n + 1}{k + 1}
\]

twice to obtain

\[
\sum_{m=1}^{n} m^2 = 2 \binom{n + 1}{3} + \binom{n + 1}{2}
\]

\[
= \frac{2(n + 1)n(n - 1)}{6} + \frac{(n + 1)n}{2}
\]

\[
= \frac{(n + 1)n}{2} \left( \frac{2(n - 1)}{3} + 1 \right)
\]

\[
= \frac{(n + 1)n}{2} \frac{2n + 1}{3}
\]

\[
= \frac{n(n + 1)(2n + 1)}{6}
\]