Problem 1. (b) The first several values of $f_0 + f_2 + \ldots + f_{2n}$ are

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f_0 + f_2 + \ldots + f_{2n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$0 + 1 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$0 + 1 + 3 = 4$</td>
</tr>
<tr>
<td>3</td>
<td>$0 + 1 + 3 + 8 = 12$</td>
</tr>
<tr>
<td>4</td>
<td>$0 + 1 + 3 + 8 + 21 = 33$</td>
</tr>
</tbody>
</table>

These numbers are all 1 less than another Fibonacci number. For example, $f_0 + f_2 + f_4 + f_6 = 34 - 1 = f_7 - 1$. As such, guess that

\[ f_0 + f_2 + \ldots + f_{2n} = f_{2n+1} - 1. \]

This holds when $n = 0$ because $f_0 = 0 = f_1 - 1$. For the induction step, assume that $n > 0$ and that

\[ f_0 + f_2 + \ldots + f_{2n-2} = f_{2n-1} - 1. \]

Then

\[
\begin{align*}
  f_0 + f_2 + \ldots + f_{2n} &= (f_0 + f_2 + \ldots + f_{2n-2}) + f_{2n} \\
  &= f_{2n-1} - 1 + f_{2n} \\
  &= f_{2n+1} - 1
\end{align*}
\]

In the last step, we used $f_{2n+1} = f_{2n-1} + f_{2n}$ by the Fibonacci recurrence.

(d) The first several values are

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f_0^2 + f_1^2 + \ldots + f_n^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0^2 = 0$</td>
</tr>
<tr>
<td>1</td>
<td>$0^2 + 1^2 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$0^2 + 1^2 + 1^2 = 2$</td>
</tr>
<tr>
<td>3</td>
<td>$0^2 + 1^2 + 1^2 + 2^2 = 6$</td>
</tr>
<tr>
<td>4</td>
<td>$0^2 + 1^2 + 1^2 + 2^2 + 3^2 = 15$</td>
</tr>
<tr>
<td>5</td>
<td>$0^2 + 1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40$</td>
</tr>
</tbody>
</table>

Each of these is a product of two consecutive Fibonacci numbers. For example, $f_0^2 + f_1^2 + \ldots + f_5^2 = 40 = (5)(8) = f_5 f_6$. As such, conjecture that

\[ f_0^2 + f_1^2 + \ldots + f_n^2 = f_n f_{n+1}. \]

This works for $n = 0$ as $f_0^2 = 0 = f_0 f_1$. Now suppose $n > 0$ and $f_0^2 + f_1^2 + \ldots + f_{n-1}^2 = f_{n-1} f_n$. Then

\[
\begin{align*}
  f_0^2 + f_1^2 + \ldots + f_n^2 &= (f_0^2 + f_1^2 + \ldots + f_{n-1}^2) + f_n^2 \\
  &= f_{n-1} f_n + f_n^2 \\
  &= f_n (f_{n-1} + f_n) = f_n f_{n+1}
\end{align*}
\]

as desired.
Problem 4. The initial conditions are easily checked. As for the recurrence
\[ f_n = f_{n-1} + f_{n-2} \]
\[ = f_{n-2} + f_{n-3} + f_{n-4} = f_{n-2} + 2f_{n-3} + f_{n-4} \]
\[ = f_{n-3} + f_{n-4} + 2(f_{n-4} + f_{n-5}) + f_{n-4} = f_{n-3} + 4f_{n-4} + 2f_{n-5} \]
\[ = f_{n-4} + f_{n-5} + 4f_{n-4} + 2f_{n-5} = 5f_{n-4} + 3f_{n-5} \]

It follows that \( f_n \equiv 3f_{n-5} \pmod{5} \). Therefore \( 5|f_n \) if and only if \( 5|3f_{n-5} \) which occurs if and only if \( 5|f_{n-5} \) (multiplying by 3 does not affect divisibility by 5). It follows by induction that \( f_n \) is divisible by 5 if and only if \( f_r \) is, where \( r \in \{0, 1, 2, 3, 4\} \) is the remainder of \( n \) by 5. Now \( 5|0 = f_0 \), but none of \( f_1, f_2, f_3, f_4 \) are divisible by 5. Therefore \( f_n \) is divisible by 5 if and only if \( n \equiv 0 \pmod{5} \).

Problem 6. First we will prove that if \( a, b \geq 1 \) then \( f_{a+b} = f_{a-1}f_b + f_a f_{b+1} \). The proof is by induction on \( a \) and \( b \). If \( a = 1 \) then
\[ f_{1+b} = 0f_b + 1f_{b+1} = f_0f_b + f_1f_{b+1} \]
and if \( b = 1 \) then
\[ f_{a+1} = f_{a-1} + f_a = f_{a-1}f_1 + f_a f_2. \]
Now suppose \( a, b \geq 2 \) and that the result holds for any pair obtained by decreasing \( a \) and/or \( b \). Then
\[ f_{a+b} = f_{a+b-1} + f_{a+b-2} \]
\[ = (f_{a-1}f_{b-1} + f_1 f_b) + (f_{a-2}f_{b-1} + f_{a-1} f_b) \]
by the induction hypothesis applied to the pairs \( a, b-1 \) and \( a-1, b-1 \). As such
\[ f_{a+b} = (f_{a-1} + f_{a-2})f_{b-1} + (f_a + f_{a-1})f_b \]
\[ = f_a f_{b-1} + f_{a+1} f_b \]
as desired.

Now suppose that \( m \) is divisible by \( n \), say \( m = kn \). Show that \( f_n | f_m \) by induction on \( k \). If \( k = 1 \) then \( n = m \) so \( f_n = f_m \). Now suppose \( k > 1 \) and the result holds for smaller \( k \). Then
\[ f_m = f_{kn} \]
\[ = f_{n+(k-1)n} \]
\[ = f_{n-1}f_{(k-1)n} + f_n f_{(k-1)n+1} \]
Therefore
\[ f_m \equiv f_{n-1}f_{(k-1)n} \pmod{f_n}. \]
But by the induction hypothesis, \( f_{(k-1)n} \) is divisible by \( f_n \). Therefore \( f_m \) is divisible by \( f_n \) as well.

Problem 17. Let \( g(x) = \sum_{n \geq 0} h_n x^n \). Then \( g(x) \) will have four factors, one for each type of fruit. Each will have the form the sum of \( x^i \) as \( i \) ranges over the legal numbers of the given fruit. Therefore
\[ g(x) = (1 + x^2 + x^4 + \ldots)(1 + x + x^2)(1 + x^3 + x^6 + \ldots)(1 + x) \]
The two infinite sums can be written in closed form
\[
g(x) = \frac{1}{1-x^2} \frac{1}{1-x^3} = \frac{1}{(1-x)(1+x)(1-x^2)}
\]
We have seen that this is the generating function for the counting numbers 1, 2, 3, …. Therefore \( h_n = n + 1 \).

**Problem 18.** Again there will be four factors, one for each term in the left hand side of the expression for \( n \). The first term must be even, the second one divisible by 5, and the fourth one must be divisible by 7. The generating function is
\[
g(x) = (1 + x^2 + x^4 + \ldots)(1 + x^5 + x^{10} + \ldots)(1 + x + x^2 + \ldots)(1 + x^7 + x^{14} + \ldots)
\]
\[
= \frac{1}{1-x^2} \frac{1}{1-x^5} \frac{1}{1-x} \frac{1}{1-x^7}
\]

**Problem 19.** We have seen that
\[
\frac{1}{(1-x)^{k+1}}
\]
is the generating function for the sequence \( \binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \ldots \). Letting \( k = 2 \)
\[
\frac{1}{(1-x)^3} = \sum_{n \geq 0} \binom{n+2}{2} x^n.
\]
Therefore
\[
\frac{x^2}{(1-x)^3} = \sum_{n \geq 0} \binom{n+2}{2} x^{n+2}
\]
\[
= \sum_{m \geq 2} \binom{m}{2} x^m
\]
\[
= \sum_{m \geq 0} \binom{m}{2} x^m
\]
The last step is justified because \( \binom{0}{2} = \binom{1}{2} = 0 \). So
\[
\frac{x^2}{(1-x)^3}
\]
is the desired generating function.