Chapter 7

Problem 26. Let \( h_n \) be the number of colorings as described in the question, and let
\[
g(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!}
\]
be its exponential generating function. We have seen that \( g(n) \) factors into four pieces, each a sum of \( x^j/j! \) where \( j \) ranges over the legal number of squares of the corresponding color. As such
\[
g(x) = \sum_{j \geq 0} \frac{x^{2j}}{(2j)!} \sum_{j \geq 0} \frac{x^j}{j!} \sum_{j \geq 0} \frac{x^{2j}}{(2j)!} \sum_{j \geq 0} \frac{x^j}{j!}
\]
By definition
\[
e^x = \sum_{j \geq 0} \frac{x^j}{j!}
\]
We have also seen that
\[
\frac{e^x + e^{-x}}{2} = \sum_{j \geq 0} \frac{x^{2j}}{(2j)!}
\]
Therefore,
\[
g(x) = \frac{e^x + e^{-x}}{2} e^x e^x + \frac{e^{-x}}{2} e^x = \left( \frac{e^{2x} + 1}{2} \right) \left( \frac{e^{2x} + 1}{2} \right) = e^{4x} + 2e^{2x} + 1
\]
\[
= \frac{1}{4} \sum_{n \geq 0} \frac{(4x)^n}{n!} + \frac{1}{2} \sum_{n \geq 0} \frac{(2x)^n}{n!} + \frac{1}{4}
\]
\[
= \frac{1}{4} + \sum_{n \geq 0} \frac{(4^{n-1} + 2^{n-1}) x^n}{n!}
\]
It follows that \( h_n = 4^{n-1} + 2^{n-1} \) for all \( n \geq 1 \). Meanwhile, \( h_0 = 1/4 + 1/4 + 1/2 = 1 \).

Problem 33. The characteristic polynomial of the recurrence is
\[
f(x) = x^3 - x^2 - 9x + 9
\]
\[
= (x - 1)(x^2 - 9)
\]
\[
= (x - 1)(x - 3)(x + 3)
\]
Therefore, the general solution to the recurrence is
\[ h_n = A(1)^n + B(3)^n + C(-3)^n. \]
Plugging in \( n = 0, 1, 2 \) yields the system of equations
\[
\begin{align*}
0 &= A + B + C \\
1 &= A + 3B - 3C \\
2 &= A + 9B + 9C
\end{align*}
\]
Subtract 9 times the first equation from the third equation to obtain
\[ 2 = -8A \]
so
\[ A = -1/4. \]
Substituting this into the first two equations leads to
\[
\begin{align*}
1/4 &= B + C \\
5/4 &= 3(B - C)
\end{align*}
\]
the solution for which is
\[ B = 1/3, C = -1/12. \]
The desired solution to the recurrence is
\[
\begin{align*}
h_n &= -1/4 + 1/3^n - 1/12(-3)^n \\
&= -3 + 4(3^n) - (-3)^n \\
&= -1 + 4(3^{n-1}) + (-3)^{n-1}
\end{align*}
\]
**Problem 38. (e)** The first few values are
\[
\begin{align*}
h_0 &= 1 \\
h_1 &= 2(1) + 1 = 3 \\
h_2 &= 2(3) + 1 = 7 \\
h_3 &= 2(7) + 1 = 15 \\
h_4 &= 2(15) + 1 = 31.
\end{align*}
\]
Each value is one less than a power of two, for example \( h_4 = 31 = 2^5 - 1. \) In general it appears that
\[ h_n = 2^{n+1} - 1. \]
This works for \( n = 0 \) since \( h_0 = 1 = 2^1 - 1. \) Now suppose \( n > 0 \) and \( h_{n-1} = 2^n - 1. \) Then
\[
\begin{align*}
h_n &= 2h_{n-1} + 1 \\
&= 2(2^n - 1) + 1 \\
&= 2^{n+1} - 2 + 1 = 2^{n+1} - 1
\end{align*}
\]
as desired.

**Problem 50.** Direct application of the definition shows that \( \{1\} \) is extraordinary, but no other subsets of \( \{1\} \) or \( \{1, 2\} \) are. Therefore \( g_1 = g_2 = 1. \) Let \( A_n \) denote the set of extraordinary subsets of \( \{1, 2, \ldots, n\}. \) Suppose \( n \geq 3. \) Clearly \( A_{n-1} \subseteq A_n \) so
\[ |A_n| = |A_{n-1}| + |B| \]
where \( B = A_n \setminus A_{n-1}. \) We will show that there is a bijection
\[ f : A_{n-2} \to B. \]
Let $S \in A_{n-2}$, that is, $S$ is an extraordinary subset of $\{1, 2, \ldots, n - 2\}$. To get $T = f(S)$ start from $S$, add one to each of its elements, and insert $n$ at the end. Symbolically

$$T = \{i + 1 : i \in S\} \cup \{n\}.$$ Clearly $|T| = |S| + 1$ (this is because everything in $S$ is at most $n - 2$, so $n$ is actually a new element of $T$). Also, the smallest element of $T$ is one more than the smallest element of $S$. Therefore $T$ is extraordinary. Also, the largest element of $T$ is $n$, so $T \in A_n$ but $T \notin A_{n-1}$. Hence $T \in B$.

Given $T \in B$, it must be the case that $n \in T$. To reverse the process of $f$, just remove $n$ and then subtract one from each of the remaining elements. Using similar reasoning as above, the result is in $A_{n-2}$. It is clear that these two processes are inverses of each other, so $f$ is a bijection. Hence $|B| = |A_{n-2}|$. By the above

$$|A_n| = |A_{n-1}| + |A_{n-2}|$$

which by definition of $g_n$ means $g_n = g_{n-1} + g_{n-2}$.

**Problem 1.** Let $h_n$ be the number of pairings as described in the problem. Give the points labels from 1 to $2n$ going around the circle. First, note that it is not important that the points are evenly spaced, or even that they lie on a circle. The result is the same as long as they are the vertices of a convex $2n$-gon.

As usual, say that $h_0 = 1$ because the empty pairing is the only way to join 0 points. Now let $n > 0$. Suppose that point 1 is joined to point $k$, with $2 \leq k \leq 2n$. This line segment separates vertices $2, 3, \ldots, k - 1$ ($k - 2$ vertices) from vertices $k + 1, k + 2, \ldots, 2n$ ($2n - k$ vertices). Hence the points in each group must be paired up to each other. This is only possible if $k - 2$ and $2n - k$ are both even, i.e. if $k$ is even. In this case, there are $h_{(k-2)/2}$ ways to pair up the first $k - 2$ vertices, and $h_{(2n-k)/2}$ ways to pair up the remaining $2n - k$. The choices are independent, so there are

$$h_{(k-2)/2}h_{(2n-k)/2} = h_{j-1}h_{n-j}$$

ways to complete the picture once 1 is joined to $k = 2j$. As $j$ can vary from 1 to $n$, we get

$$h_n = h_0h_{n-1} + h_1h_{n-2} + \ldots + h_{n-1}h_0$$

The numbers $h_n$ satisfy the same recurrence as the Catalan numbers $C_n$, with the same initial value $h_0 = C_0 = 1$. Therefore $h_n = C_n$ for all $n \geq 0$.

**Problem 2.** Let $A$ be the set of 2-by-$n$ arrays with the stated properties, and let $B$ be the set of length $2n$ Dyck paths. In the notation of class, $B$ consists of length $2n$ strings of $+$’s and $-$’s with the property that there are the same number $(n)$ of each, and reading from left to right you never see more $-$’s than $+$’s. If $T \in A$, then each of $1, 2, \ldots, 2n$ appear exactly once in $T$. We can define a string $s = s_1s_2 \cdots s_{2n}$ by $s_i = +$ if $i$ is in the first row of $T$ and $s_i = -$ if $i$ is in the second row of $T$. Clearly $s$ has an equal number of $+$’s and $-$’s. Let $s_1s_2 \cdots s_k$ be a prefix of $s$, say with $a$ $+$’s and $b$ $-$’s, where $a + b = k$. Let

$$T = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{bmatrix}.$$
The a+'s in $s_1 \cdots s_k$ correspond to the first $a$ entries of the first row, so $x_{1a} \leq k < x_{1,a+1}$. Similarly, $x_{2b} \leq k < x_{2,b+1}$. If $b > a$, then

$$k < x_{1,a+1} \leq x_{1b} < x_{2b} \leq k$$

a contradiction. Every prefix of $s$ has at least as many +’s as −’s, so $s \in B$. So the operation producing $s$ from $T$ defines a function $F : A \rightarrow B$.

If $F(T_1) = F(T_2)$, then $T_1$ and $T_2$ have the same collection of entries in each row. The rows have to be in increasing order, so it follows that $T_1 = T_2$. Therefore $F$ is injective.

Lastly, let $s \in B$. Define $T$ with entries $x_{ij}$ where $x_{1j}$ is the position of the $j$th + in $s$ and $x_{2j}$ is the position of the $j$th − in $s$. Clearly the elements of each row of $T$ are increasing. Moreover, since $s \in B$, the $j$th − always comes after the $j$th +, so $x_{1j} < x_{2j}$ for all $j$. Therefore $T \in A$ and $F(T) = B$. So, $F$ is surjective.

We have shown that $F$ is a bijection. Therefore $|A| = |B| = C_n$.

**Problem 4. (a)** There are 6 variables, so the identification is with a triangulation of a 7-gon. Label all the edges except for the bottom one from 1 to 6. Each pair of parentheses, except for the outermost pair, encompasses a nonempty proper subset of the variables. For each such, draw a diagonal that separates the corresponding sides from the remaining sides. For example, the parentheses $(a_2 \times a_3)$ gives rise to the short diagonal that cuts out sides 2 and 3. The result is the four diagonals pictured below, which do in fact give a triangulation.

[Diagram of a 7-gon with labeled vertices and diagonals]