Chapter 8

**Problem 7.** The portion of the difference table that can be easily computed is

\[
\begin{array}{cccc}
1 & -1 & 3 & 10 \\
-2 & 4 & 7 & \\
6 & & 3 & \\
-3 & & & \\
\end{array}
\]

The 0th diagonal begins \(c_0 = 1\), \(c_1 = -2\), \(c_2 = 6\) and \(c_3 = -3\). Since \(h_n\) is a degree 3 polynomial, \(c_n = 0\) for \(n > 3\).

In general, we have proven

\[ h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \ldots + c_d \binom{n}{d} \]

and

\[ \sum_{k=0}^{n} h_k = c_0 \binom{n+1}{1} + c_1 \binom{n+1}{2} + \ldots + c_d \binom{n+1}{d+1} \]

where \(d\) is the degree of the polynomial. In our case, \(d = 3\), and the \(c_i\) are as above, so

\[ h_n = \binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} = 1 - 2n + 3n(n - 1) - \frac{n(n - 1)(n - 2)}{2} \]

and

\[ \sum_{k=0}^{n} h_k = \binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4} = n + 1 - (n + 1)n + (n + 1)n(n - 1) - \frac{(n + 1)n(n - 1)(n - 2)}{8} \]

**Problem 12.** (c) By definition, \(S(n, n-1)\) counts the number of ways to partition \(\{1, 2, \ldots, n\}\) into \(n - 1\) nonempty subsets. The only way to accomplish this is to put two elements together in one box (i.e. subset), and give each of the remaining \(n - 2\) their own box. Conversely, for any choice of two elements, it is possible to put them together and leave everything else separate to get a legal partition. So \(S(n, n-1)\) equals the number of pairs of elements from 1 through \(n\), namely \(S(n, n-1) = \binom{n}{2}\).

**Problem 13.** For concreteness, let \(X = \{1, 2, \ldots, p\}\) and \(Y = \{1, 2, \ldots, k\}\). Recall that

\[ k!S(p, k) \]

equals the number of ordered partitions of \(X\) into \(k\) non-empty subsets \(A_1, A_2, \ldots, A_k\). Given a surjection \(f : X \to Y\), it is natural to associate the partition \(A_1 = f^{-1}(1), \ldots, A_k = f^{-1}(k)\).
\[ A_k = f^{-1}(k). \] In words, \( A_i \) is the set of elements of \( X \) that map to \( i \). This forms a partition since each element of \( X \) is mapped to exactly one \( i \in \{1, 2, \ldots, k\} \). Each \( A_i \) is nonempty because \( f \) is surjective so each \( i \) has at least one element mapping to it.

It remains to show that the operation above taking a surjection \( f \) to a partition \( A_1, \ldots, A_k \) of \( X \) into nonempty subsets is reversible. Indeed, given a partition \( A_1, \ldots, A_k \), one can define a function \( f \) by \( f(j) = i \) where \( A_i \) is the unique set containing \( j \). Since the \( A_i \) are all nonempty, \( f \) will be surjective. It is clear that this procedure is inverse to the one from before. So the number of surjections \( f \) also equals \( k!S(p, k) \).

**Problem 19.**

(a) The expression \( s(n, 1) \) counts the number of partitions of \( \{1, 2, \ldots, n\} \) into a single circular permutation. In other words, \( s(n, 1) \) equals the number of circular permutations of \( \{1, 2, \ldots, n\} \) which we have seen equals \( (n-1)! \).

(b) The expression \( s(n, n-1) \) counts the number of partitions of \( \{1, 2, \ldots, n\} \) into \( n-1 \) nonempty circular permutations. By the pigeonhole principle the only possibility is that one of the permutations contains 2 elements and the other \( n-2 \) contain 1 each. There are \( \binom{n}{2} \) choices of the pair of elements to put together. There is no further freedom because there is just one circular permutation of 2 elements. So \( s(n, n-1) = \binom{n}{2} \).

**Problem 29.** Each such partition has the form \( n = 2 + \ldots + 2 + 1 + \ldots + 1 \) where the number of 2’s equals \( a \) and the number of 1’s equals \( b \). Here, \( a \) and \( b \) must be nonnegative integers satisfying \( 2a + b = n \). It follows that
\[
0 \leq a \leq \frac{n}{2}.
\]

Put another way, the possible values of \( a \) are \( 0, 1, 2, \ldots, \lfloor n/2 \rfloor \). Each gives a single solution, so there are \( \lfloor n/2 \rfloor + 1 \) total solutions
\[
(a, b) \in \{(0, n), (1, n - 2), (2, n - 4), \ldots, (\lfloor n/2 \rfloor, n - 2\lfloor n/2 \rfloor)\}.
\]

This also gives the total number of partitions with each part at most 2.

1. **Chapter 14**

**Problem 14.** There are \( 2^3 = 8 \) total 2-colorings which can be grouped by equivalence as follows:

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>RR</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>W</td>
</tr>
<tr>
<td>W</td>
<td>WR</td>
<td>RR</td>
</tr>
</tbody>
</table>

Hence there are 4 nonequivalent colorings. Given 3 colors there are \( 3^3 = 27 \) total colorings, arranged as follows:
Note for example that $R \ W B$ and $R \ BW$ are equivalent because they are reflections of each other. So there are 10 nonequivalent colorings.