The Mean Value Theorem and the Extended Mean Value Theorem

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0.1 The MVT

Recall the Extreme Value Theorem (EVT) from class: If the function f is defined and continuous on a closed bounded interval $[a, b]$ then there is some point $c \in [a, b]$ where it takes on its maximum value $M = f(c)$ and some point $d \in [a, b]$ where it takes on its minimum value $m = f(d)$. Thus

$$M = f(c) \geq f(x) \geq f(d) = m$$

for all $x \in [a, b]$.

Theorem 1 *Fermat's Theorem. (Not his last one but a very useful observation that is easy to prove.) Suppose the function f is defined and continuous on a closed bounded interval $[a, b]$ and takes on an extreme value (either its maximum M or its minimum m) at an interior point c of the interval, so $a < c < b$. If the derivative $f'(c)$ exists, then $f'(c) = 0$.*

PROOF: To be definite, we assume that $f(c) = M$. (The proof for the case $f(c) = m$ is virtually the same.) By definition

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}. \tag{1}$$

Since $f(c) = M$ is a maximum, it follows that for $\Delta x > 0$ we have

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0,$$

since the numerator is nonpositive and the denominator is positive. On the other hand, for $\Delta x < 0$ we have

$$\frac{f(c+\Delta x) - f(c)}{\Delta x} \geq 0,$$

since the numerator is nonpositive and the denominator is negative. However by assumption the limit (1) exists and the same value is obtained as $\Delta x \to 0$ through postitive or negative values. Thus we must have $f'(c) = 0$. Q.E.D.

Let $a < b$ be finite numbers.

Theorem 2 *Mean Value Theorem (MVT). Suppose the function f is defined and continuous on a closed bounded interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a point c, $a < c < b$, such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

PROOF: Consider the secant line ℓ that connects the endpoints $(a, f(a))$ and $(b, f(b)$ on the graph of the function $y = f(x)$. Using the point-slope equation for a line, we see that the equation for the secant line is

$$\ell : \quad y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

For each $x \in [a, b]$ let $g(x)$ be the directed distance between the curve $f(x)$ and the line $y(x)$:

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}x - \frac{f(b) - f(a)}{b - a}a \right].$$

Note that $g(a) = g(b) = 0$ and

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Since g vanishes at the endpoints and g is continuous on $[a, b]$ by the EVT there must be some interior point $c \in (a, b)$ such that $g(c)$ is an extreme value of g. By Fermat's theorem $g'(c) = 0$, which means

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Q.E.D.

0.2 The EMVT

Theorem 3 *Extended Mean Value Theorem (EMVT). Suppose $f(x), g(x)$ are functions such that*

1. *f, g are defined and continuous on the closed bounded interval $[a, b]$, $a < b$.*
2. *f, g are differentiable on the open interval (a, b).*
3. *$g'(x) \neq 0$ for all $x \in (a, b)$.*

Then there exists a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

NOTE: If $g(x) = x$, then this is just the statement of the Mean Value Theorem (MVT).

PROOF: For the MVT we considered the function

$$F(x) = f(x) - L(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) - f(a).$$

We observed that $F(a) = F(b) = 0$ and that there must be a relative extremum of F at some $c \in (a, b)$. Then by the Fermat Theorem we must have $F'(c) = 0$. But

$$F'(x) = f'(x) - \left(\frac{f(b) - f(a)}{b - a}\right).$$

For the EMVT we apply the same procedure to the function

$$F(x) = f(x) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)(g(x) - g(a)) - f(a).$$

Again $F(a) = F(b) = 0$ and there must be a relative extremum of F at some $c \in (a, b)$. By the Fermat Theorem and assumption 1. we must have $F'(c) = 0$. Thus

$$F'(c) = f'(c) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(c) = 0.$$

or

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)},$$

since $g'(c) \neq 0$. Q.E.D.

0.3 A $\frac{0}{0}$ form of the L'Hospital Rule

Theorem 4 *Let $f(x)$ and $g(x)$ be differentiable in an open neighborhood N containing $x = a$, (but not necessarily at $x = a$), $f'(x), g'(x)$ continuous in the same neighborhood, and suppose $g'(x) \neq 0$ in N. Suppose*

$$\lim_{x \to a} f(x) = 0, \qquad \lim_{x \to a} g(x) = 0.$$

Then

$$\lim_{h \to a} \frac{f(x)}{g(x)} = \lim_{h \to a} \frac{f'(x)}{g'(x)},$$

where the equality is meant in any one of the senses

1. *Both limits exist and are equal.*
2. *Both limits diverge to $+\infty$.*
3. *Both limits diverge to $-\infty$.*

If the right hand limit fails to exist, the rule is inconclusive.

Corollary 1 *The Rule is also true if the limits are right hand $(x \to a+)$ or left-hand $(x \to a-)$.*

PROOF OF THE RULE: Since $\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0$, we can extend the domains of f, g to $x = a$, if necessary, by defining $f(a) = g(a) = 0$. Then f and g are continuous at a. Now let x be in the neighborhood N, with $x > a$. Then the EMVT applies to the interval $[a, x]$ and there is a $y \in (a, x)$ such that

$$\frac{f'(y)}{g'(y)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Thus

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(y)}{g'(y)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}.$$

if the last limit exists or diverged to $\pm\infty$. Similarly, if $x < a$ we apply the EMVT to the interval $[x, a]$ and get the same result for the limit as $x \to a-$. Q.E.D.

0.4 A $\frac{\infty}{\infty}$ form of the L'Hospital Rule

The proofs of the $\frac{\infty}{\infty}$ forms of the L'Hospital Rule are a little trickier. We will prove a right-hand limit version; the proof of the left-hand limit case is virtually identical. If both left and right-hand limits exist and are equal, then the two-sided limit exists.

Theorem 5 *Suppose*

1. $f(x)$ *and* $g(x)$ *are differentiable in an open interval* (a, b), *and* $f'(x), g'(x)$ *are continuous in the same neighborhood*
2. $g'(x) \neq 0$ *on* (a, b)
3. $\lim_{x\to a+} f(x) = +\infty, \qquad \lim_{x\to a+} g(x) = +\infty$

Then if

$$\lim_{h\to a+} \frac{f'(x)}{g'(x)} = L,$$

we also have

$$\lim_{h\to a+} \frac{f(x)}{g(x)} = L,$$

where the equality is meant in any one of the senses

1. *Both limits exist and are equal.*
2. *Both limits diverge to* $+\infty$.
3. *Both limits diverge to* $-\infty$.

The test is inconclusive if the right hand limit fails to exist.

PROOF: Let $a < x < y < b$ and apply the EMVT to the interval $[x, y]$. Then there is a $\xi \in (x, y)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(x)}{g(x)} \frac{1 - \frac{f(y)}{f(x)}}{1 - \frac{g(y)}{g(x)}}. \tag{2}$$

Now we let *both* x and y approach a, but with x making the approach more rapidly than y, so that

$$\lim \frac{f(y)}{f(x)} = \lim \frac{g(y)}{g(x)} = 0.$$

This can be done because of assumption 3. Thus

$$\lim \frac{1 - \frac{g(y)}{g(x)}}{1 - \frac{f(y)}{f(x)}} = 1$$

and, since $\frac{f'(x)}{g'(x)}$ is continuous,

$$\lim \frac{f'(\xi)}{g'(\xi)} = \lim_{h \to a+} \frac{f'(x)}{g'(x)} = L.$$

It follows from (2) that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim \frac{f'(\xi)}{g'(\xi)} \lim \frac{1 - \frac{g(y)}{g(x)}}{1 - \frac{f(y)}{f(x)}} = L \cdot 1 = L.$$

Q.E.D.