Limit Rules and Derivative Rules

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0.1 Limit Rules

We will make use of the following limit rules in this course, without any rigorous proof. For most of the functions we encounter, these rules should seem intuitively obvious to you. Let $f(x)$ and $g(x)$ be functions with limits at $x = a$:

$$\lim_{x \to a} f(x) = L, \qquad \lim_{x \to a} g(x) = M.$$

The following rules hold.

1. SUM RULE: The limit of a sum is the sum of the limits.

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M.$$

2. PRODUCT RULE: The limit of a product is the product of the limits.

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = LM.$$

3. QUOTIENT RULE: The limit of a quotient is the quotient of the limits (with a caveat). If $M \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}.$$

0.2 Some Simple Limits

1. Limit of a constant function $f(x) = c$:

$$\lim_{x \to a} c = c.$$

2. Limit of the function $f(x) = x$:

$$\lim_{x \to a} x = a.$$

3. Limit of the polynomial function $P(x) = c_0 + c_1 x + \cdots + c_n x^n$:

$$\lim_{x \to a} P(x) = c_0 + c_1 a + \cdots + c_n a^n = P(a).$$

0.3 Differentiability Implies Continuity

Suppose $f(x)$ is differentiable at $x = a$, i.e., the following limit exists:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Then $f(x)$ is also continuous at $x = a$, i.e.,

$$\lim_{x \to a} f(x) = f(a).$$

PROOF: Using the product rule for limits we have

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{[f(x) - f(a)](x - a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a) = f'(a) \cdot 0 = 0.$$

Thus $\lim_{x \to a} f(x) = f(a)$.

0.4 Derivative Rules

Let $f(x)$ and $g(x)$ be differentiable at $x = a$, i.e.,

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a), \qquad \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a).$$

The following rules hold.

1. SUM RULE: The derivative of a sum is the sum of the derivatives:

   $$k(x) = f(x) + g(x) \implies k'(a) = f'(a) + g'(a).$$

   PROOF: Using the sum rule for limits we have

   $$k'(a) = \lim_{h\to 0} \frac{k(a+h) - k(a)}{h} = \lim_{h\to 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h}$$

   $$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = f'(a) + g'(a).$$

2. PRODUCT RULE: If $k(x) = f(x)g(x)$ then k is differentiable at $x = a$ and

   $$k'(a) = f'(a)g(a) + f(a)g'(a).$$

   PROOF: We have to compute the limit

   $$k'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

   Note that

   $$f(a+h)g(a+h) - f(a)g(a) = f(a+h)\left[g(a+h) - g(a)\right] + g(a)\left[f(a+h) - f(a)\right].$$

   Thus

   $$k'(a) = \lim_{h\to 0} \frac{f(a+h)\left[g(a+h) - g(a)\right] + g(a)\left[f(a+h) - f(a)\right]}{h}$$

   $$= \lim_{h\to 0} f(a+h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} + g(a) \cdot \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

by the limit rules for sums and products. Since f is differentiable at a, it is also continuous there, so $\lim_{h\to 0} f(a+h) = f(a)$. Thus

$$\begin{aligned} k'(a) &= f(a) \cdot \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} + g(a) \cdot \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= f(a)g'(a) + f'(a)g(a). \end{aligned}$$

3. RECIPROCAL RULE: If $g(x)$ is differentiable at a with $g(a) \neq 0$, then the reciprocal $k(x) = \frac{1}{g(x)}$ is differentiable at a with derivative

$$k'(a) = -\frac{g'(a)}{g^2(a)}.$$

PROOF:

$$\begin{aligned} k'(a) &= \lim_{h\to 0} \left(\frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} \right) = -\lim_{h\to 0} \left(\frac{g(a+h)-g(a)}{g(a)g(a+h)h} \right) \\ &= \frac{-1}{g(a)} \cdot \lim_{h\to 0} \frac{1}{g(a+h)} \cdot \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}, \end{aligned}$$

by the limit rule for products. Now since $g(x)$ is continuous at $x = a$ with $g(a) \neq 0$, it follows from the quotient rule for limits that $1/g(x)$ is also continuous at $x = a$. Thus

$$k'(a) = -\frac{g'(a)}{g^2(a)}.$$

4. QUOTIENT RULE: If $f(x)$ and $g(x)$ are differentiable at a with $g(a) \neq 0$, then the quotient $\ell(x) = \frac{f(x)}{g(x)}$ is differentiable at a with derivative

$$\ell'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

PROOF: Setting $k(x) = 1/g(x)$ we have (from the product and reciprocal rules for derivatives)

$$\ell'(a) = \left(\frac{f(x)}{g(x)} \right)' \Bigg|_{x=a} = (f(x)k(x))' \Big|_{x=a}$$

$$= f'(a)k(a) + f(a)k'(a) = \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g^2(a)} = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$