6709 Alternating Series

Definition. A infinite series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ with $b_n > 0$ is called an alternating series.

The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2n+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \ldots$ is an example of an alternating series. The word alternating refers to the fact that the signs of the terms alternate between + and − . We have the following theorem which can be used to show that an alternating series is convergent.

Theorem. The Alternating Series Theorem. Given the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ with $b_n > 0$.

If (1) $\lim_{n\to\infty} b_n = 0$ and (2) $b_{n+1} < b_n$,

then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

Note that the hypothesis, the phase which follows the word "if" in the statement of the theorem, contains two parts. This means that we will always have to show two things to be true in order to conclude that an alternating series is convergent. This theorem can not be used to show that an alternating series diverges. The proof of this theorem is fairly easy, but long. Note that this is the first theorem we will discussed in detail where the terms of the series can be negative as well as positive.

Example 1. Show that the following alternating series converges:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+2}{n^2+4}.$$

Solution. The clue that this is an alternating series is the $(-1)^{n-1}$. We can also write out a few terms of this series

$$(3/5) - (4/8) + (5/13) - (6/20) + (7/29) - (8/40) + \ldots$$

Note that the + and − signs alternate in this series. This is the feature that indicates that it is an alternating series. As soon as we are certain that this is an alternating series we try to show that it converges using the Alternating Series Theorem. In order to do this we replace b_n by $\frac{n+2}{n^2+4}$ in the Alternating Series Theorem. We get the following true statement:

(a) If $\lim_{n\to\infty} \dfrac{n+2}{n^2+4} = 0$ and $\dfrac{(n+1)+2}{(n+1)^2+4} < \dfrac{n+2}{n^2+4}$,

then $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+2}{n^2+4}$ converges.

The hypothesis of this true statement is the clause that comes after the word "if" and before the comma. The conclusion is the clause that comes after the work "then". The hypothesis of this statement is:

$$\lim_{n\to\infty} \frac{n+2}{n^2+4} = 0 \text{ and } \frac{(n+1)+2}{(n+1)^2+4} < \frac{n+2}{n^2+4}.$$

The conclusion is:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+2}{n^2+4} \text{ converges.}$$

In order to conclude that the conclusion is true we must first show that the hypothesis is true.

Note that there are two parts to the hypothesis of this statement and that the two parts are connected by the word "and". Because the parts are connected by the word "and", we must show that both parts of the hypothesis are true in order to say that the conclusion is true. First part:

$$(b) \qquad \lim_{n\to\infty} \frac{n+2}{n^2+4} = \lim_{n\to\infty} \frac{1+2/n}{n+4/n} = 0.$$

For this series $b_n = \frac{n+2}{n^2+4}$. We also need b_{n+1}. We get b_{n+1} by replacing n with $n+1$ in the formula for b_n. We must show $b_{n+1} < b_n$. The inequality $\dfrac{n+3}{n^2+2n+5} < \dfrac{n+2}{n^2+4}$ is true if $(n+3)(n^2+4) < (n+2)(n^2+2n+5)$ which is true if $n^3+3n^2+4n+12 < n^3+4n^2+9n+10$ which is true if $2 < n^2+5n$ which is clearly true. Therefore, it is true that

$$(c) \qquad \frac{n+3}{n^2+2n+5} < \frac{n+2}{n^2+4}.$$

Since both parts of the hypothesis have been shown to be true it follows that the conclusion is true. The true statements (a),(b), and (c) cause us to conclude that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+2}{n^2+4} \text{ converges}$$

is a true statement.

We have shown that this series converges, but we have no idea what the sum of the series is. The Alternating Series Theorem give us no help in finding the sum of the series.

Example 2. Show that the following alternating series converges:

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{\sqrt{2k+5}}.$$

Solution. This is an alternating series. We can replace b_k in the Alternating Series Theorem by any numbers and we will always get a true statement. First, since $b_k = \frac{1}{\sqrt{2k+5}}$, then $b_{k+1} = \frac{1}{\sqrt{2k+7}}$. Replace b_k by $\frac{1}{\sqrt{2k+5}}$ in the Alternating Series Theorem and we get the following true statement.

If $\lim_{k\to\infty}\frac{1}{\sqrt{2k+5}} = 0$ and $\frac{1}{\sqrt{2k+7}} < \frac{1}{\sqrt{2k+5}}$, then $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\sqrt{2k+5}}$ converges.
The hypothesis of this implication is:

$$\lim_{k\to\infty}\frac{1}{\sqrt{2k+5}} = 0 \text{ and } \frac{1}{\sqrt{2k+7}} < \frac{1}{\sqrt{2k+5}}.$$

The conclusion is

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\sqrt{2k+5}} \text{ converges.}$$

We must show that the hypothesis is true. Clearly

$$\lim_{k\to\infty}\frac{1}{\sqrt{2k+5}} = 0.$$

Using properties of inqualities $\frac{1}{\sqrt{2k+7}} < \frac{1}{\sqrt{2k+5}}$ is true if $\sqrt{2k+5} < \sqrt{2k+7}$ which is true if $2k+5 < 2k+7$ which is true if $5 < 7$. Therefore,

$$\frac{1}{\sqrt{2k+7}} < \frac{1}{\sqrt{2k+5}}.$$

Since both parts of the hypothesis are true it follows that the conclusion is true. From the true statements (a),(b), and (c) taken together we conclude that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{2k+1}} \text{ converges.}$$

We know that this series converges, but we do not know its sum. Because we do not know what the exact sum of this series is we often try to find an approximate value for the exact sum.

Let us consider the problem of getting an approximate value for the sum of an alternating series. Consider the simple alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

It is easy to show that this alternating series converges using the Alternating Series Theorem. Let us assume we have done that part. Once we know that the series is convergent we might ask: What is its exact sum? We can always approximate the exact sum by adding a few terms together. For example, the sum of 8 terms is

$$1 - (1/4) + (1/9) - (1/16) + (1/25) - (1/36) + (1/49) - (1/64) = 0.8156.$$

We feel fairly certain that if we were to add more terms we would get a more accurate approximation. For example, the sum of 16 terms is

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots - \frac{1}{256} = 0.8206.$$

The question is: how accurate is this approximation? How accurate is the sum of the first 100 terms as an approximation? In the general discussion

of this question we use the following notation. We can write any series as the sum of two parts as follows:

$$\sum_{k=1}^{\infty}(-1)^{k-1}b_k = \sum_{k=1}^{N}(-1)^{k-1}b_k + \sum_{k=N+1}^{\infty}(-1)^{k-1}b_k.$$

Here we are thinking of the entire series as expressed as a sum of the first N terms of the series plus the sum of the rest of the terms. We can use any counting number for N. When $N = 20$ this would be

$$\sum_{k=1}^{\infty}(-1)^{k-1}b_k = \sum_{k=1}^{20}(-1)^{k-1}b_k + \sum_{k=21}^{\infty}(-1)^{k-1}b_k.$$

Here we have expressed the series as the sum of the first 20 terms plus the sum of all the other terms starting with the 21st term. We have already used the notation

$$S_N = \sum_{k=1}^{N}(-1)^{k-1}b_k.$$

We will also use the notation

$$R_N = \sum_{k=N+1}^{\infty}(-1)^{k-1}b_k.$$

Using this notation

$$\sum_{k=1}^{\infty}(-1)^{k-1}b_k = S_N + R_N.$$

We have broken the sum of the entire series into two parts. The first sum S_N is a finite sum and so can be calculated. The second sum R_N can be thought of as the remainder of the series. Also since

$$\left[\sum_{k=1}^{\infty}(-1)^{k-1}b_k\right] - S_N = R_N$$

we see that R_N is the error we make when we approximate the whole sum $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ with just the finite sum $S_N = \sum_{k=1}^{N}(-1)^{k-1}b_k$. The following theorem gives us a maximum value or bound for the error term R_N.

Theorem. Alternating Series Estimation Theorem. Given the alternating series $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ such that

(1) $\lim_{k\to\infty} b_k = 0$ (2) $b_{k+1} < b_k$,

then

$$\left|R_N\right| = \left|\sum_{k=N+1}^{\infty}(-1)^{k-1}b_k\right| < b_{N+1}.$$

Recall that in order to have an alternating series we must have $b_k > 0$. Let us use this theorem to estimate the error resulting when the finite sum

$$1-(1/4)+(1/9)-(1/16)+\ldots-(1/256)=0.8206$$

is used to estimate the whole sum $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$. Breaking up the series as was done in general earlier, we get

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}=\sum_{k=1}^{16}\frac{(-1)^{k-1}}{k^2}+\sum_{k=17}^{\infty}\frac{(-1)^{k-1}}{k^2}$$

The difference between the exact value $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$ and the approximate value $\sum_{k=1}^{16}\frac{(-1)^{k-1}}{k^2}$ is $\sum_{k=17}^{\infty}\frac{(-1)^{k-1}}{k^2}$. Substituting $N=16$ and $b_k=1/k^2$ into $\left|\sum_{k=N+1}^{\infty}(-1)^{k-1}b_k\right| < b_{N+1}$, which is the formula of the Alternating Series Estimation Theorem, we get

$$\left|\sum_{N=17}^{\infty}(-1)^{k-1}\frac{1}{k^2}\right| < \frac{1}{(17)^2}=\frac{1}{289}.$$

The error is less than $1/289 = 0.0035$. The actual value is between $0.8206 - 0.0035 = 0.8171$ and $0.8206 + 0.0035 = 0.8241$.

Example 3. Assume that $\lim_{k\to\infty} \frac{3k+2}{2k^3+5} = 0$ and $\frac{3(k+1)+2}{2(k+1)^3+5} < \frac{3k+2}{2k^3+5}$ are both true. Use the Alternating Series Estimation Theorem to find an upper bound for

$$\left|\sum_{k=15}^{\infty}(-1)^k \frac{3k+2}{2k^3+5}\right| \text{ and } \left|\sum_{k=50}^{\infty}(-1)^k \frac{3k+2}{2k^3+5}\right|.$$

Solution. Substituting $b_k = \frac{3k+2}{2k^3+5}$ in the Alternating Series Estimation Theorem, the hypothesis is:

$$\lim_{k\to\infty} \frac{3k+2}{2k^3+5} = 0 \text{ and } \frac{3(k+1)+2}{2(k+1)^3+5} < \frac{3k+2}{2k^3+5}.$$

We are given that this is true. This means that the Alternating Series Estimation Theorem applies to the series. Therefore, we are able to conclude using first $N = 14$ and then $N = 49$ that

$$\left|\sum_{k=15}^{\infty}(-1)^k \frac{3k+2}{2k^3+5}\right| < \frac{3(15)+2}{2(15)^3+5} = \frac{47}{6755} < .007.$$

$$\left|\sum_{k=50}^{\infty}(-1)^k \frac{3k+2}{2k^3+5}\right| < \frac{3(50)+2}{2(50)^3+5} = \frac{152}{250005} = .0006$$

Example 4. Given $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k+2}{k^4+8}$. Suppose we approximate the exact sum using the finite sum $\sum_{k=1}^{20}(-1)^{k-1}\frac{k+2}{k^4+8}$, what is the maximum possible error we make according to the Alternating Series Estimation Theorem?

Solution. We first show that both parts of the hypothesis of the Alternating Series Estimation Theorm are true by showing that

(1) $\lim_{k\to\infty} \frac{k+2}{k^4+8} = 0$

(2) $\dfrac{k+3}{(k+1)^4+8} < \dfrac{k+2}{k^4+8}$.

Let us assume that this has been done. Since both parts of the hypothesis are true, we can apply the Alternating Series Estimation Theorem to find a bound on the error.

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k+2}{k^4+8} = \sum_{k=1}^{20}(-1)^{k-1}\frac{k+2}{k^4+8} + \sum_{k=21}^{\infty}(-1)^{k-1}\frac{k+2}{k^4+8}.$$

The remainder or error term is

$$\sum_{k=21}^{\infty}(-1)^{k-1}\frac{k+2}{k^4+8}.$$

Replacing N by 20 in the Alternating Series Estimation Theorem, we have

$$\left|\sum_{k=21}^{\infty}(-1)^{k-1}\frac{k+2}{k^4+8}\right| < \frac{21+2}{(21)^4+8} = \frac{23}{194,489} = .000118$$

The error is less than 0.00012. Note that $N = 20$ is $N+1 = 21$.

Example 5. Given the alternating series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^3}$. Suppose we want to approximate the exact sum of the whole series using a finite sum. How many terms do we need in the finite sum in order to be sure that the error in the approximation is less than 10^{-4}?

Solution. First, we really need to make sure that the alternating series $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$ converges. This is the same as showing that the hypothesis of the Alternating Series Estimation Theorem is true. The hypothesis of the Alternating Series Estimation Theorem is

(1) $\lim_{k\to\infty}\frac{1}{k^3} = 0$ and (2) $\dfrac{1}{(k+1)^3} < \dfrac{1}{k^3}$.

Clearly $\lim_{k\to\infty} k^{-3} = 0$. We have $(k+1)^{-3} < k^{-3}$ is true if $k^3 < (k+1)^3$ which is true if $k < k+1$. The hypothesis is true. We conclude that the

conclusion of the estimation theorem is true. Using the general value N the Alternating Series Estimation Theorem now allows us to conclude that the following is true:

$$\left|\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}-\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k^3}\right|=\left|\sum_{k=N+1}^{\infty}\frac{(-1)^{k-1}}{k^3}\right|<\frac{1}{(N+1)^3}.$$

This inequality says that for any value of N the error is less than $1/(N+1)^3$. For this problem we need to choose a value of N such that

$$\frac{1}{(N+1)^3}<10^{-4}.$$

This is the same as

$$\begin{aligned}(N+1)^3&>10^4\\ N+1&>10^{4/3}\\ N&>20.54\end{aligned}$$

We make the fraction $1/(N+1)$ smaller by making N larger. Also N must be a whole number. The smallest value of N we can choose is $N=21$. The error in approximating $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$ with the finite sum $\displaystyle\sum_{k=1}^{21}\frac{(-1)^{k-1}}{k^3}$ is less than 10^{-4} or is correct to 4 decimal places. Note that we usually consider the phrase "correct to 4 decimal places" to mean "$|R_N|<10^{-4}$".

We now discuss some new theorems. We easily see that $\displaystyle\lim_{n\to\infty}\frac{3n+2}{5n+7}=\frac{3}{5}$. Also $(-1)^{n-1}$ alternately takes the values $1,-1,1,-1,\ldots$. It follows that for large n the fraction $\displaystyle(-1)^n\frac{3n+2}{5n+7}$ alternately takes the values close to $3/5$ and $-3/5$. Therefore

$$\lim_{n\to\infty}(-1)^n\frac{3n+2}{5n+7}\text{ does not exist.}$$

Using this same reasoning we easily see that the following lemma is true.

Lemma. If $\displaystyle\lim_{n\to\infty}b_n=L$ and $L\neq 0$, then $\displaystyle\lim_{n\to\infty}(-1)^n b_n$ does not exist.

Recall the following theorem from the section on definition of infinite series.

Theorem. If $\lim_{n\to\infty}(-1)^n b_n$ does not exist, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is divergent.

Combining this theorem and the lemma we get the following theorem.

Divergence theorem for Alternating Series. If $\lim_{n\to\infty} b_n = L$ and $L \neq 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is divergent.

Example 6. Show that the following Alternating Series is divergent using the Divergence Theorem for Alternating Series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+3}{5n+8}.$$

Solution. Replacing b_n in The Divergence Theorem for Alternating Series with $b_n = \dfrac{n+3}{5n+8}$, we get the true statement:

(a) If $\lim_{n\to\infty}\dfrac{n+3}{5n+8} = \dfrac{1}{5}$ and $1/5 \neq 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n+3}{5n+8}$ is divergent.

Certainly (b) $1/5 \neq 0$. Also (c) $\lim_{n\to\infty}\dfrac{n+3}{5n+8} = \lim_{n\to\infty}\dfrac{1+3/n}{5+8/n} = \dfrac{1}{5}$. Since the hypothesis of the statement is true it follows that the conclusion is true. The three true statements (a),(b), and (c) taken together cause us to conclude that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+3}{5n+8} \text{ is divergent.}$$

We only have two theorems about the convergence of alternating series. One is the Alternating Series Theorem which we use to show that an alternating series is convergent. The other is the Divergence Theorem for Alternating Series which we use to show that an alternating series is divergent as we did in the Example.

Example 7. Is the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+3}{n^2+1}$ convergent or divergent?

Solution. We have only two theorems about alternating series. In order to apply either theorem we must find $\lim_{n\to\infty}\frac{2n+3}{n^2+1}$ since the value of this limit is part of the hypothesis of both theorems. Start by finding this limit.

$$\lim_{n\to\infty}\frac{2n+3}{n^2+1}=\lim_{n\to\infty}\frac{2+3/n}{n+1/n}=0.$$

A limit of zero is part of the hypothesis of the Alternating Series Theorem. This means that we want to use the Alternating Series Divergence Theorem. Let $b_n=\frac{2n+3}{n^2+1}$ in the Alternating Series Theorem and we get the following true statement:

(a) If $\lim_{n\to\infty}\frac{2n+3}{n^2+1}=0$ and $\frac{2n+5}{n^2+2n+2}<\frac{2n+3}{n^2+1}$, then $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+3}{n^2+1}$ is convergent.

(b) $\lim_{n\to\infty}\frac{2n+3}{n^2+1}=\lim_{n\to\infty}\frac{2/n^2+3/n}{1+1/n^2}=\frac{0}{1}=0$ was shown above.

Now $\frac{2n+5}{n^2+2n+2}<\frac{2n+3}{n^2+1}$ is true if $2n^3+5n^2+2n+5<2n^3+7n^2+10n+6$ which is true if $0<2n^2+8n+1$. Therefore,

(c)
$$\frac{2n+5}{n^2+2n+5}<\frac{2n+3}{n^2+1}.$$

The true statements (a),(b), and (c) taken together cause us to conclude that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+3}{n^2+1}$ is convergent.

Exercises

1. Given that the following statements are all true, what other statement are you able to conclude is also true?

(a) If $\dfrac{5n+13}{n^2+2n+5} < \dfrac{5n+8}{n^2+4}$ and $\lim\limits_{n\to\infty} \dfrac{5n+8}{n^2+4} = 0$, then $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{5n+8}{n^2+4}$ converges.

(b) $\lim\limits_{n\to\infty} \dfrac{5n+8}{n^2+4} = 0$

(c) $\dfrac{5n+13}{n^2+2n+5} < \dfrac{5n+8}{n^2+4}$ is true for all n.

2. Given that the following two statements are both true, what other statement are we able to conclude is also true?

(a) If $\lim\limits_{n\to\infty} \dfrac{2n+3}{5n+8} = \dfrac{2}{5}$ and $(2/5) \neq 0$, then $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{2n+3}{5n+8}$ is divergent.

(b) $\lim\limits_{n\to\infty} \dfrac{2n+3}{5n+8} = \dfrac{2}{5}$.

3. a) Replace b_n by $\dfrac{2n+1}{n(n+1)}$ in the Alternating Series Theorem.

b) Show that $\lim\limits_{n\to\infty} \dfrac{2n+1}{n(n+1)} = 0$

c) Show that $\dfrac{2n+3}{(n+1)(n+2)} < \dfrac{2n+1}{n(n+1)}$.

d) Considering the true statements in parts a,b, and c taken together what can you conclude?

4. a) Replace b_n by $\dfrac{1}{\sqrt{4n^2+3}}$ in the Alternating Series Theorem.

b) Look at your answer in part (a). Show that the hypothesis of the implicaiton which is part (a) is true.

c) Considering the true statements in parts (a) and (b) what are you able to conclude?

5. Show that the Alternating Series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k^2+5}$ is convergent.

6. Use Alternating Series Divergence Theorem to show that an alternating series is divergent.

a) Replace b_n in Theorem 2 by $\frac{3n+2}{5n+4}$.

b) Show that $\lim_{n\to\infty} \frac{3n+2}{5n+4} = \frac{3}{5}$.

c) You have two true statements here. What conclusion can you reach?

7. Is the following alternating series convergent or divergent? Justify your answer by applying a theorem.

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{5k+3}{k^2+4}$$

8. Given that $\lim_{k\to\infty} \frac{3k+5}{7k^3+8} = 0$ and that $\frac{3(k+1)+5}{7(k+1)^3+8} < \frac{3k+5}{7k^3+8}$, find the maximum value of each of the following using the Alternating Series Estimation Theorem.

a) $\left| \sum_{k=25}^{\infty} (-1)^{k-1} \frac{3k+5}{7k^3+8} \right|$

b) $\left| \sum_{k=100}^{\infty} (-1)^{k-1} \frac{3k+5}{7k^3+8} \right|$

c) $\left| \sum_{k=1}^{\infty} (-1)^{k-1} \frac{3k+5}{7k^3+8} \right|$

d) $\left| \sum_{k=1}^{\infty} (-1)^{k-1} \frac{3k+5}{7k^3+8} - \sum_{k=1}^{42} \frac{3k+5}{7k^3+8} \right|$

e) $\left|\sum_{k=1}^{\infty}(-1)^{k-1}\frac{3k+5}{7k^3+8}-\sum_{k=1}^{100}\frac{3k+5}{7k^3+8}\right|$

9. a) Show that $\lim_{k\to\infty} k^{-4}=0$.

b) Show that $(k+1)^{-4}<k^{-4}$ for all k.

c) Find a maximum value for each of the following using the Alternating Series Estimation Theorem:

i) $\left|\sum_{k=30}^{\infty}\frac{(-1)^{k-1}}{k^4}\right|$ ii) $\left|\sum_{k=63}^{\infty}\frac{(-1)^{k-1}}{k^4}\right|$

iii) $\left|\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^4}-\sum_{k=1}^{30}\frac{(-1)^{k-1}}{k^4}\right|$ $\left|\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^4}-\sum_{k=1}^{59}\frac{(-1)^{k-1}}{k^4}\right|$

d) Find the smallest value of N such that

i) $\left|\sum_{k=N+1}^{\infty}\frac{(-1)^{k-1}}{k^4}\right|<10^{-2}$

ii) $\left|\sum_{k=N+1}^{\infty}\frac{(-1)^{k-1}}{k^4}\right|<10^{-6}$

10. Consider the alternating series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k+4}{5k^3+8}$. Given that the hypothesis of the Alternating Series Estimation Theorem are satisfied.

a) Find a maximum value for $\left|\sum_{k=50}^{\infty}(-1)^{k-1}\frac{k+4}{5k^3+8}\right|$.

b) Suppose we approximate the infinite sum $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k+4}{5k^3+8}$ using the finite sum $\sum_{k=1}^{30}(-1)^{k-1}\frac{k+4}{5k^3+8}$, what is the maximum possible error we make

according to the Alternating Series Estimation Theorem.

c) Find a maximum value for the following using the Alternating Series Estimation Theorem.

$$\left|\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k+4}{5k^3+8}-\sum_{k=1}^{39}(-1)^{k-1}\frac{k+4}{5k^3+8}\right|.$$

11. Given that the hypothesis of the Alternating Series Estimation Theorem are satisfied for the series $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{5k^3+8}$ find the smallest value of N such that:

a) $\left|\sum_{k=N+1}^{\infty}\frac{(-1)^{k-1}}{5k^3+8}\right| < 10^{-4}$.

b) $\left|\sum_{k=N+1}^{\infty}\frac{(-1)^{k-1}}{5k^3+8}\right| < 10^{-8}$.