6711 Ratio Test

One of the easiest ways to tell if an infinite series is convergent is to use the ratio test. The down side of the ratio test is that it often fails to tell us if the series is convergent or if it is divergent. In many problems we are unable to reach any conclusion after applying the ratio test.

Theorem. The Ratio Test. Given the infinite series $\sum_{n=1}^{\infty} a_n$, let L denote the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

1. If $0 \le L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $L > 1$ or $L = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If $L = 1$, the test gives no information about convergence. Test fails.

We could now prove that the ratio test theorem is a true statement. We would only need the idea of absolute convergence and theorems which we have already stated. However, the proof is long and so we choose to omit it.

Example 1. Is the series $\sum_{n=1}^{\infty} \frac{n(2n+5)}{2^n}$ convergent or divergent? Justify your answer.

Solution. Apply the ratio test. The first step is to find the ratio a_{n+1}/a_n. In order to find the ratio we need a_n and a_{n+1}. We obtain a_{n+1} by replacing n with $n+1$ in the expression for a_n. For this problem

$$a_n = \frac{n(2n+5)}{2^n} \quad \text{and} \quad a_{n+1} = \frac{(n+1)(2n+7)}{2^{n+1}}.$$

The ratio is found by dividing a_{n+1} by a_n. In order to divide the fraction a_{n+1} by the fraction a_n we invert the fraction a_n and multiply. The ratio is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(2n+7)}{2^{n+1}} \div \frac{n(2n+5)}{2^n}$$
$$= \frac{(n+1)(2n+7)}{2^{n+1}} \cdot \frac{2^n}{n(2n+5)} = \frac{(n+1)(2n+7)}{2n(2n+5)}.$$

When we simplified we used the fact that $2^n/2^{n+1} = 1/2$. We did this by subtracting powers. Next find L which is the limit of the ratio. Since all these numbers are positive we do not need to worry about taking the absolute value.

$$L = \lim_{n\to\infty} \frac{(n+1)(2n+7)}{2n(2n+5)} = \lim_{n\to\infty} \frac{(n+1)(2n+7)(1/n^2)}{(2n)(2n+5)(1/n^2)}$$
$$= \lim_{n\to\infty} \frac{(1+1/n)(2+7/n)}{2(2+5/n)} = \frac{1}{2}.$$

Note that we multiplied the numerator and denominator both by the same number $(1/n^2)$. This did not change the value of the fraction. Replacing L by $1/2$ and a_n by $\dfrac{n(2n+5)}{2^n}$ in Part 1 of the Ratio Test, we get the following true statement:

If $1/2 < 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{n(2n+5)}{2^n}$ converges.

Clearly $1/2 < 1$ is true. Therefore, we are able to conclude that "$\displaystyle\sum_{n=1}^{\infty} \frac{n(2n+5)}{2^n}$ converges" is true.

Example 2. Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{2n^2+5}$ convergent or divergent?

Solution. We want to use the ratio test. First, find the ratio which is a_{n+1}/a_n. Note that

$$a_n = \frac{(-1)^{n-1}3^n}{2n^2+5} \text{ and } a_{n+1} = \frac{(-1)^n 3^{n+1}}{2(n+1)^2+5}.$$

Remember that in order to divide by the fraction a_n we invert and multiply. Inverting a_n and multiplying we get the ratio

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^n 3^{n+1}}{2n^2+4n+7} \cdot \frac{2n^2+5}{(-1)^{n-1}3^n} = -\frac{3(2n^2+5)}{2n^2+4n+7}.$$

When simplifying this fraction we used the fact that $(-1)^n/(-1)^{n-1} = -1$ and $3^{n+1}/3^n = 3$ which we get by subtracting powers. Next we take the absolute value and then find the limit. We get

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{3(2n^2+5)}{2n^2+4n+7} \\ &= \lim_{n\to\infty} \frac{6+15/n^2}{2+4/n+7/n^2} = \frac{6}{2} = 3. \end{aligned}$$

Replacing L by 3 and a_n by $\dfrac{(-1)^{n-1}3^n}{2n^2+5}$ in the second part of the Ratio Test Theorem, we get the following true statement:

If $3 > 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{2n^2+5}$ is divergent.

Clearly $3 > 1$. Therefore, we are able to conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{2n^2+5}$ is divergent.

Example 3. Apply the ratio test to the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n+5}{7n^2+10}$.

Solution. First, we have

$$a_n = \frac{(-1)^{n-1}(3n+5)}{7n^2+10} \text{ and } a_{n+1} = \frac{(-1)^n(3n+8)}{7n^2+14n+17}.$$

The ratio is a_{n+1} divided by a_n. In order to divide we invert and multiply. After inverting the fraction a_n we compute the ratio as follows:

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^n(3n+8)}{7n^2+14n+17} \cdot \frac{7n^2+10}{(-1)^{n-1}(3n+5)} = \frac{-(3n+8)(7n^2+10)}{(7n^2+14n+17)(3n+5)}.$$

Take the absolute value and then multiply the numerator and denominator by $1/n^3$. This does not change the value of the fraction. Taking the limit, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(3+8/n)(7+10/n^2)}{(7+14/n+17/n^2)(3+5/n)} = 1.$$

We get $L=1$. When $L=1$, the ratio test gives no information. We must use some other theorem to determine if this series converges. For this series we could use the Alternating Series Theorem to show that it converges.

Exercises

1. a) Consider the infinite series $\sum_{n=1}^{\infty}\frac{2n+5}{(n+1)2^n}$. Find a_n and a_{n+1} for this series.

b) Find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

c) Substitute into the appropriate part of the Ratio Test Theorem. What conclusions do you reach?

2. a) Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^n}{2n^2+5n+4}$. Find a_n and a_{n+1} for this series.

b) Find the limit of the absolute value of the ratio.

c) Substitute into the appropriate part of the Ratio Test Theorem. What are you able to conclude?

3. a) Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n^2+5}$. Find a_n and a_{n+1} for this series.

b) Find the limit of the absolute value of the ratio.

c) Substitute into the appropriate part of the Ratio Test Theorem. What are you able to conclude?

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)2^n}{(5n+3)3^n}$. Use the Ratio Test Theorem to determine if this series is convergent or divergent. This means substitute into the appropriate part of the ratio test. Show all steps.

5. Consider the infinite series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2n^2+3}$. Apply the Ratio Test to this series. Is this series convergent or divergent? Show all steps.

6713 Power Series

An infinite series of the form $\sum_{n=0}^{\infty} b_n x^n$ is called a power series. The numbers b_n are constants depending on n whereas x is an independent variable. The variable x can be replaced with any real number. Consider the power series

$$\sum_{n=0}^{\infty} \frac{nx^n}{n^2+4}$$

where x is a variable. We get somewhat different series by replacing x with different numbers. If we replace x by 2 we get the series $\sum_{n=0}^{\infty} \frac{n2^n}{n^2+4}$. If we replace x by $3/4$, we get the series $\sum_{n=0}^{\infty} \frac{n(3/4)^n}{n^2+4}$. Since we are allowed to replace x by different numbers the series may converge when we replace x by certain numbers and diverge when we replace x by other numbers. This means that when given a power series we ask ourselves the question: for exactly which values of x does this series converge and for exactly which values does it diverge?

A remark on the notation for power series. Most series considered up to this point have been written using $\sum_{n=1}^{\infty}$. The first term considered is $n = 1$. On the other hand, power series are usually written with $\sum_{n=0}^{\infty}$. For power series the first term considered is the $n = 0$ term. This is the usual way to write power series since this causes the powers of x to be of the form x^n. When testing a series for convergence, it makes no difference whether we start n with $n = 0$ or $n = 1$. The tests for convergence involve $\lim_{n\to\infty}$ and do not depend on what happens with the small values of n.

Example 1. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{2^n x^n}{n^2+1}$ converge?

Solution. In order to determine the values of x for which a power series converges we always apply the Ratio Test. The first step in the ratio test is to find the ratio a_{n+1}/a_n. For this power series

$$a_n = \frac{2^n x^n}{n^2+1} \text{ and } a_{n+1} = \frac{2^{n+1} x^{n+1}}{n^2+2n+2}.$$

Recall that in order to divide we invert and multiply. The ratio is

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} x^{n+1}}{n^2+2n+2} \cdot \frac{n^2+1}{2^n x^n} = \frac{2x(n^2+1)}{n^2+2n+2}.$$

We simplified using the fact that $x^{n+1}/x^n = x$ and $2^{n+1}/2^n = 2$. Recall that we must take the absolute value of the ratio when we find L. The number L in the Ratio Test is given by

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2x(n^2+1)}{n^2+2n+2} \right|$$

We can factor $|x|$ out in front of the limit sign since $|x|$ does not depend on n.

$$= 2|x| \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = 2|x|.$$

Since $L = 2|x|$ we replace L with $2|x|$ and a_n with $\dfrac{2^n x^n}{n^2+1}$ in the Ratio Test. This gives us the following two true statements:

1) If $2|x| < 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2+1}$ converges.

2) If $2|x| > 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2+1}$ diverges.

Note that "$2|x| < 1$" is the same as "$x > -1/2$ and $x < 1/2$". The statement "$2|x| > 1$" is the same as "$x < -1/2$ or $x > 1/2$". Recall that when $L = 1$ there is no conclusion in the Ratio Test. For this series $L = 1$ is $2|x| = 1$, that is, $x = -1/2$ or $x = 1/2$. We do not know from the Ratio Test if this series converges or if it diverges when $x = -1/2$ and $x = 1/2$. The values $x = -1/2$ and $x = 1/2$ are known as *end points* for the power series. We will not try to determine if a power series converges or if it diverges at its end points. We could use other theorems and show that this particular power series converges for $x = -1/2$ and $x = 1/2$. We will not do this. Given a power series we will only apply the ratio test in order to find for what values of x it converges. For a power series suppose $L = |x|/R$ then R is called the radius of convergence of the power series. For this series we say that the radius of convergence is $R = 1/2$.

Example 2. For what values of x does the following power series converge?

$$\sum_{n=0}^{\infty} \frac{(n+1)^2 x^n}{5^n}$$

Solution. We always apply the Ratio Test and only the Ratio Test to determine the values of x for which a power series converges. For this power series

$$a_n = \frac{(n+1)^2 x^n}{5^n} \text{ and } a_{n+1} = \frac{(n+2)^2 x^{n+1}}{5^{n+1}}$$

We need to find a_{n+1} divided by a_n. Recall that to divide we invert and multiply. After inverting the fraction a_n, the ratio is

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)^2 x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(n+1)^2 x^n} = \frac{x}{5} \frac{(n+2)^2}{(n+1)^2}.$$

We used the fact that $x^{n+1}/x^n = x$ and $5^n/5^{n+1} = 1/5$. Do not forget to take absolute value when finding L.

$$L = \lim_{n\to\infty} \left| \frac{x}{5} \frac{(n+2)^2}{(n+1)^2} \right| = \frac{|x|}{5} \lim_{n\to\infty} \frac{(1+2/n)^2}{(1+1/n)^2} = \frac{|x|}{5}.$$

We can factor $|x|/5$ out in front of the limit sign since $|x|/5$ does not depend on n. Replacing L with $|x|/5$ and a_n with $\dfrac{(n+1)^2 x^n}{5^n}$ in the ratio test, we get the true statements

1. If $\dfrac{|x|}{5} < 1$, then $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2 x^n}{5^n}$ converges.

2. If $\frac{|x|}{5} > 1$, then $\sum_{n=0}^{\infty} \frac{(n+1)^2 x^n}{5^n}$ diverges.

Note that $\frac{|x|}{5} < 1$ is the same as $-5 < x < 5$, which is the same as $-5 < x$ and at the same time $x < 5$. Also note that $\frac{|x|}{5} > 1$ is the same as "$x < -5$ or $x > 5$". The end points for this power series are $x = -5$ and $x = 5$. We do not try to decide if the power series is convergent for these two values of x. The radius of convergence for this power series is 5. The interval of convergence is $-5 < x < 5$.

Example 3. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Solution. For this power series

$$a_n = \frac{x^n}{n!} \text{ and } a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

The ratio is

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x(n!)}{(n+1)!}$$

The definition of factorial says that $4! = 1 \cdot 2 \cdot 3 \cdot 4$ and $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$. This means that $6! = 6(5!)$ and $10! = 10(9!)$. In general $(n+1)! = (n+1)(n!)$. This can be rewritten as

$$\frac{1}{n+1} = \frac{n!}{(n+1)!}$$

Using this we reduce the fraction as follows:

$$\frac{a_{n+1}}{a_n} = \frac{x(n!)}{(n+1)!} = \frac{x}{n+1}.$$

Taking the limit of this ratio we find L:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

This limit is zero for all x. Replacing L by 0 and a_n by $x^n/n!$ in the first part of the ratio test, we get the true statement.

If $0 < 1$, then $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

From this we conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all values of x.

Exercises

1. a) Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2n+3)x^n}{(n^2+1)2^n}$. Find a_n and a_{n+1} for this series.

b) Find $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

c) Substitute into the Ratio Test Theorem. What conclusions do you reach?

2. (a) Consider the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n^2+4)x^n}{3n+5}$. Find a_n and a_{n+1} for this series.

b) Next find the ratio a_{n+1}/a_n and then find the limit of the absolute value of the ratio, that is, find L.

c) Substitute into the Ratio Test Theorem. What conclusions do you reach?

3. a) Consider the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Find a_n and a_{n+1} for this series.

b) Find the ratio $\dfrac{a_{n+1}}{a_n}$.

c) Find the limit of the absolute value of the ratio.

d) Substitute into the Ratio Test Theorem. What conclusions do you reach?

4.(a) Consider the power series $\sum_{n=0}^{\infty}(-1)^n(n+1)3^n x^n$. Find a_n and a_{n+1} for this series.

b) Find the limit of the absolute value of the ratio.

c) Substitute into the Ratio Test Theorem. What is your conclusion?

6715 Taylor's Series and Polynomials

We are now going to discuss: given a function $f(x)$ find the Taylor's series for $f(x)$. In order to find the Taylor's series for $f(x)$, the function $f(x)$ must have an infinite number of derivatives. You will notice that the Taylor's series is also a power series.

Definition. The formula for the Taylor's series for a function $f(x)$ about $a = 0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots$$

Example 1. Find the Taylor's series about $a = 0$ (also called the Maclaurian series) for the function $f(x) = e^{5x}$.

Solution. We need to find $f^{(n)}(0)$ in order to substitute for it in the formula for Taylor's series. This means we need to find all the derivatives of the function $f(x)$. The first few derivatives are:

$$\begin{array}{ccc} f'(x) = 5e^{5x} & f''(x) = 5^2 e^{5x} & f'''(x) = 5^3 e^{5x} \\ f^{(4)}(x) = 5^4 e^{5x} & f^{(5)}(x) = 5^5 e^{5x} & f^{(6)}(x) = 5^6 e^{5x}. \end{array}$$

We need $f^{(n)}(x)$. The hardest part of finding the Taylors series for a function $f(x)$ is to find a general formula for the general nth derivative of $f(x)$. We do this by looking at the pattern of the first half dozen or so derivatives. For this example note that the 4th derivative has 5^4 as a factor, the 5th derivative has 5^5, the 6th derivative has 5^6. From this pattern we conclude that for any nth derivative we would have 5 to nth power or 5^n as the coefficient of e^{5x}. Therefore,

$$f^{(n)}(x) = 5^n e^{5x}.$$

This says that the 15th derivative is given by $f^{(15)}(x) = 5^{15} e^{5x}$. We need the nth derivative evaluated when $x = 0$.

$$f^{(n)}(0) = 5^n e^0 = 5^n.$$

$$\frac{f^{(n)}(0)}{n!} = \frac{5^n}{n!}$$

We substitute this into $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ and find Taylor's series for e^{5x} is

$$e^{5x} = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n.$$

We also have the question: for what values of x does this Taylor's series for e^{5x} converge? Note that this Taylor's series is a power series. We could determine for what values of x this series converges by using the Ratio test. Indeed we can determine for what values of x any Taylor's series converges by applying the ratio test. However, in a complete discussion of Taylor's series we do not need to apply the ratio test to determine for what values of x the Taylor's series converges. In fact as part of a complete discussion of Taylor's series we find out not only for what values of x the series converges but also the sum of the series. Knowing the sum of the series can be very helpful. The ratio test never tells us the sum of a power series. We will not do a complete discussion of Taylor's series because it is to time consuming.

In general finding the Taylor's series for a given function is a difficult task. The difficult part is to find a formula for $f^{(n)}(x)$. For this reason, we will not find many Taylor's series by directly using the formulas given above. However, we do need to know a few basic Taylor's series. As part of the work in finding these Taylor's series we could also find the sum of the series and the values of x for which the series converge. The Taylor's series for a few common functions are given below.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \quad \text{for } |x| < 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots, \text{ for all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ for all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \text{ for all } x$$

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \text{ for } |x| < 1.$$

Note that these Taylor's series are power series. We will use the terms Taylor's series and power series as though they had the same meaning. In some more advanced discussions they are used to have very slightly different meanings. In these discussions the term "Taylor's series" is used to indicate that the series was originally found using the derivative formulas.

Definition. The Taylor polynomial $T_n(x)$ of order n of the function $f(x)$ about the value $a = 0$ is given by

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n.$$

The Taylor polynomial $T_n(x)$ of order n for $f(x)$ is the terms up to x^n of the Taylor's series for $f(x)$. The Taylor polynomial $T_3(x)$ of order 3 for $f(x)$ is the terms up to x^3 of the Taylor's series for $f(x)$. The general formula for $T_3(x)$ is

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

The Taylor polynomial $T_5(x)$ of order 5 for $f(x)$ is the terms up to x^5 of the Taylor's series for $f(x)$. The general formula for $T_5(x)$ is

$$T_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5.$$

Example 2. The Taylor polynomial $T_4(x)$ of degree 4 for $f(x) = e^x$ is

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

We obtain this polynomial by looking at the Taylor's series for e^x and copying down the terms up to x^4.

Example 3. The Taylor polynomial $T_6(x)$ of degree 6 for $f(x) = \cos x$ is

$$T_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

We get the polynomial $T_6(x)$ by looking at the Taylor's series for $\cos x$ and copying down the terms up to x^6.

Example 4. Find $T_4(x)$, the Taylor polynomial of degree 4 about $x = 0$, for $f(x) = \sin 5x + \cos 4x$.

Solution. We are not given the Taylor's series for this function. In order to find this Taylor polynomial we must use the derivative formulas. We start by finding the first four derivatives. They are

$$\begin{aligned} f(x) &= \sin 5x + \cos 4x \\ f'(x) &= 5\cos 5x - 4\sin 4x \\ f''(x) &= -25\sin 5x - 16\cos 4x \\ f'''(x) &= -125\cos 5x + 64\sin 4x \\ f^{(4)}(x) &= 625\sin 5x + 256\cos 4x \end{aligned}$$

Next, we evaluate these derivatives when $x = 0$.

$$\begin{array}{ccc} f(0) = 1 & f'(0) = 5 & f''(0) = -16 \\ f'''(0) = -125 & f^{(4)}(0) = 256 & \end{array}$$

Next divide each derivative by the appropriate factorial:

$$\frac{f''(0)}{2} = -8, \quad \frac{f'''(0)}{3!} = -\frac{125}{6} \quad \text{and} \quad \frac{f^{(4)}(0)}{4!} = \frac{256}{24} = \frac{32}{3}.$$

Substituting into the formula for the Taylor polynomial, we get the polynomial for $f(x) = \sin 5x + \cos 4x$.

$$T_4(x) = 1 + 5x - 8x^2 - \frac{125}{6}x^3 + \frac{32}{3}x^4.$$

Example 5. Find $T_5(x)$, the Taylor polynomial of degree 5, for $f(x) = \ln(9 + x)$.

Solution. Since we do not know the Taylor's series for $\ln(9 + x)$ we must use the derivative formulas to find $T_5(x)$. In order to substitute into the formula for $T_5(x)$ we must find the first five derivatives of $f(x) = \ln(9 + x)$.

$$\begin{array}{ll} f'(x) = (9 + x)^{-1} & f^{(4)}(x) = -6(9 + x)^{-4} \\ f''(x) = (-1)(9 + x)^{-2} & f^{(5)}(x) = 24(9 + x)^{-5} \\ f'''(x) = 2(9 + x)^{-3} & \end{array}$$

Next, we evaluate the given function and these derivatives when $x = 0$.

$$
\begin{array}{ll}
f(0) = \ln 9 & f'''(0) = 2(9)^{-3} \\
f'(0) = 9^{-1} & f^{(4)}(0) = -6(9)^{-4} \\
f''(0) = -9^{-2} & f^{(5)}(0) = 24(9)^{-5}.
\end{array}
$$

Next divide the derivatives by the appropriate factorial:

$$
\begin{array}{ll}
\dfrac{f''(0)}{2!} = \dfrac{-9^{-2}}{2} = -\dfrac{1}{162} & f^{(4)}(0) = \dfrac{-6(9)^{-4}}{4!} = -\dfrac{1}{26244} \\
f'''(0) = \dfrac{2(9)^{-3}}{3!} = \dfrac{1}{2187} & f^{(5)}(0) = \dfrac{24(9)^{-5}}{5!} = \dfrac{1}{295,245}.
\end{array}
$$

Substituting into the formula for the Taylor polynomial, we get

$$
T_5(x) = \ln 9 + \frac{x}{9} - \frac{x^2}{162} + \frac{x^3}{2187} - \frac{x^4}{26244} + \frac{x^5}{295,245}.
$$

So far we have discussed two methods for finding the Taylor polynomial of a function. First method, if we are real lucky the polynomial we want to find is just the first few terms of a known Taylor's series. Second method, we can take the derivatives of the function and substitute into the general formula for the Taylor polynomial.

There are also other methods for finding Taylor polynomials which we are now going to discuss. We can also find the Taylor polynomial of a function by manipulating the polynomial of a related function.

Example 6. Find the Taylor polynomial of degree 5 for $\sin 5x$ and for $\sin(x/3)$. Start with the known Taylor polynomial for $\sin x$.

Solution. The Taylor polynomial of degree 5 for $\sin x$ is

$$
\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}.
$$

We find this polynomial by copying the terms up to x^5 in the power series for $\sin x$. Replacing x with $5x$ we obtain the polynomial of degree 5 for $\sin 5x$:

$$
\sin 5x \approx 5x - \frac{(5x)^3}{6} + \frac{(5x)^5}{120}.
$$

The Taylor polynomial $T_5(x)$ for $\sin 5x$ is

$$T_5(x) = 5x - \frac{125x^3}{6} + \frac{3125x^5}{120}.$$

Replacing x with $x/3$ we obtain the polynomial of degree 5 for $\sin(x/3)$.

$$\begin{aligned} \sin\frac{x}{3} &\approx \frac{x}{3} - \frac{(x/3)^3}{6} + \frac{(x/3)^5}{120} \\ &\approx \frac{x}{3} - \frac{x^3}{162} + \frac{x^5}{29160}. \end{aligned}$$

Example 7. Find the Taylor polynomial of degree 4 for $f(x) = (4+x)^{-1}$.

Solution. We start by finding the Taylor polynomial of degree 4 for $(1+x)^{-1}$. It is the terms up to x^4 of power series for $(1+x)^{-1}$ which is given above. The Taylor polynomial is

$$(1+x)^{-1} \approx 1 - x + x^2 - x^3 + x^4.$$

Replacing x with $x/4$ we get the polynomial of degree 4 for $[1+(x/4)]^{-1}$.

$$\begin{aligned} \frac{1}{1+(x/4)} &\approx 1 - \frac{x}{4} + \frac{x^2}{4^2} - \frac{x^3}{4^3} + \frac{x^4}{4^4} \\ \frac{4}{4+x} &\approx 1 - \frac{x}{4} + \frac{x^2}{16} - \frac{x^3}{64} + \frac{x^4}{256} \end{aligned}$$

Dividing both sides by 4 we get

$$\frac{1}{4+x} \approx \frac{1}{4} - \frac{x}{16} + \frac{x^2}{64} - \frac{x^3}{256} + \frac{x^4}{1024}.$$

Fourth method. We are now going to discuss a fourth method for finding a Taylor polynomial. This method involves differentiating the function and differentiating the corresponding polynomial.

Example 8. Find the Taylor polynomial for $(4+x)^{-2}$.

Solution. Note that $\frac{d}{dx}(4+x)^{-1} = (-1)(4+x)^{-2}$. We can also differentiate the polynomial for $(4+x)^{-1}$. Differentiating both sides gives

$$-\frac{1}{(4+x)^2} \approx -\frac{1}{16} + \frac{x}{32} - \frac{3x^2}{256} + \frac{x^3}{256}$$
$$\frac{1}{(4+x)^2} \approx \frac{1}{16} - \frac{x}{32} + \frac{3x^2}{256} - \frac{x^3}{256}$$

Note that this is $T_3(x)$ the Taylor polynomial for $(4+x)^{-2}$ of degree 3.

Fifth method. Finally, we are going to discuss a fifth method for finding a Taylor polynomial. This method involves integrating the function and integrating the corresponding polynomial.

Example 9. Find the Taylor polynomial for $\ln(4+x)$.

Solution. Note that $\int (4+x)^{-1}dx = \ln(4+x) + C$. We start with

$$(4+x)^{-1} \approx \frac{1}{4} - \frac{x}{16} + \frac{x^2}{64} - \frac{x^3}{256} + \frac{x^4}{1024},$$

which was found above. First,

$$\int \left[\frac{1}{4} - \frac{x}{16} + \frac{x^2}{64} - \frac{x^3}{256} + \frac{x^4}{1024}\right] dx$$
$$= C + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{3(64)} - \frac{x^4}{4(256)} + \frac{x^5}{5(1024)}.$$

This tells us that

$$\ln(4+x) \approx C + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{3(64)} - \frac{x^4}{4(256)} + \frac{x^5}{5(1024)}.$$

In order to find the constant of integration C let us substitute $x = 0$ into both sides. We get

$$\ln 4 = C$$

The value of the constant of integration is $C = \ln 4$.

$$\ln(4+x) \approx \ln 4 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{192} - \frac{x^4}{1024} + \frac{x^5}{5120}.$$

We can also find the value of C as follows. The expression

$$C+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{3(64)}-\frac{x^4}{4(256)}+\frac{x^5}{5(1024)}$$

is the Taylor polynomial $T_5(x)$ for $\ln(5+x)$. We do not know the constant term C in this polynomial. However, the constant term in the Taylor polynomial for $f(x)$ is always given by $f(0)$. It follows that $C=f(0)$ or $C=\ln(4+0)=\ln 4$.

This is $T_5(x)$, the Taylor polynomial of degree 5, for $\ln(4+x)$.

Essentially all the Taylor polynomials we have discussed so far have been in powers of $(x-0)^n$. It is possible to use a center $a \neq 0$. The Taylor polynomial $T_4(x)$ about the general number a rather than $a=0$ is

$$T_4(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4$$

Example 10. Find the Taylor polynomial $T_4(x)$ for $f(x)=\sqrt{x}$ about $a=4$.

Solution. First, note that $a \neq 0$. The general formula for the Taylor polynomial of order n when $a \neq 0$ is given above. We need to find the first 4 derivatives of $f(x)=\sqrt{x}$.

$$f'(x)=\frac{1}{2}x^{-1/2} \qquad f''(x)=\frac{1}{2}(-\frac{1}{2})x^{-3/2}$$
$$f'''(x)=\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^{-5/2} \qquad f^{(4)}(x)=\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})x^{-7/2}$$

We need to evaluate the given function and these derivatives for $a=4$.

$$\begin{aligned}
f(4)&=\sqrt{4}=2\\
f'(4)&=(1/2)(4)^{-1/2}=1/4\\
f''(4)&=-(1/4)(4)^{-3/2}=-1/32\\
f'''(4)&=(3/8)(4)^{-5/2}=3/256\\
f^{(4)}(4)&=-(15/16)(4)^{-7/2}=-(15/2048)
\end{aligned}$$

Dividing by the appropriate factorial, we get

$$\frac{f^{(2)}(4)}{2} = -\frac{1}{64} \qquad \frac{f'''(4)}{3!} = \frac{1}{512}$$
$$\frac{f^{(4)}(4)}{4!} = -\frac{5}{16,384}.$$

Substituting into the formula we get the Taylor polynomial

$$T_4(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4.$$

The general Taylor polynomial of order 1 for $f(x)$ about $x = a$ is

$$T_1(x) = f(a) + f'(a)(x-a).$$

This polynomial is also known as the linearization of $f(x)$ at $x = a$.

Example 11. Let $f(x) = (4x+9)^{3/2}$. Find the Taylor polynomial of order 1 for $f(x)$ about $x = 10$.

Solution. We need to find $f'(x)$.

$$f'(x) = (3/2)(4x+9)^{1/2}(4) = 6(4x+9)^{1/2}.$$

$$f(10) = (49)^{3/2} = 343, \text{ and } f'(10) = 6(49)^{1/2} = 42$$

The Taylor polynomial is

$$T_1(x) = 343 + 42(x-10).$$

The Taylor polynomial of order one of $f(x)$ about $x = a$ is also called the linearization of $f(x)$ at $x = a$. The linearization of $f(x) = (4x+9)^{3/2}$ at $x = 10$ is

$$L(x) = 343 + 42(x-10).$$

Exercises

1. By looking at the appropriate power series find the Taylor polynomial $T_4(x)$ for each of the following functions:

a) $\cos x$ b) $\arctan x$ c) $\sin x$

2. Start with an appropriate easy to find Taylor polynomial and by substituting into it find the Taylor polynomial $T_5(x)$ for each of the following functions:

a) $\dfrac{1}{1-x^2}$ b) $\cos(5x)$ c) e^{-x^2}

3. Use the derivative formulas for finding a Taylor polynomial to find the Taylor polynomial $T_4(x)$ for each of the following functions:

a) $f(x) = \sin 3x + \cos 5x$ b) $f(x) = \sqrt{1+2x}$

4. Start with the Taylor polynomial of degree 5 for $(1-x)^{-1}$ and find the Taylor polynomial of degree 5 for $f(x) = (5+x)^{-1}$.

5. Start with the Taylor polynomial of degree 6 for $\arctan x$ and find the Taylor polynomial of degree 5 for $f(x) = (1+x^2)^{-1}$ using differentiation.

6. The Taylor polynomial of degree 4 for $(4+x)^{-1/2}$ about $x = 0$ is

$$\frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \frac{5}{2048}x^3 + \frac{35}{65,536}x^4.$$

Starting with this polynomial find the Taylor polynomial $T_5(x)$ of degree 5 about $x = 0$ for the function $f(x) = (4+x)^{1/2}$.

b) Starting with the given Taylor polynomial, find the Taylor polynomial $T_3(x)$ about $x = 0$ for the function $g(x) = (4+x)^{-3/2}$.

7. The Taylor polynomial of degree 4 for $(8+x)^{2/3}$ about $x = 0$ is

$$4 + \frac{x}{3} - \frac{x^2}{144} + \frac{x^3}{2592} - \frac{7x^4}{248832}..$$

Find the Taylor polynomial $T_4(x)$ of degree 4 about $x = 0$ for $(8+x)^{5/3}$.

8. Let $f(x) = x^{3/2}$. Find the linearization of $f(x)$ at $x = 64$.

9. Find the Taylor polynomial $T_4(x)$ for $f(x) = \sqrt{x+1}$ about $a = 8$.