Name: ______________________________

Section: ______________________________

Math 1571H. Final Exam December 15, 2005

There are a total of 235 points on this exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit.

Problem	Score
1.	
2.	
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11.	
12.	
13.	
14.	
Total:	

Problem 1 *The vector equation of a curve is given by*

$$\mathbf{R}(t) = 2\sec t\ \mathbf{i} + 3\tan t\ \mathbf{j}$$

where $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

a. (5 points) *What is the locus of points* (x, y) *on the graph of this curve? Be as specific as possible.*

Solution: Set $\mathbf{R}(t) = x\ \mathbf{i} + y\ \mathbf{j}$. *Then* $\sec t = x/2$, $\tan t = y/3$ *so*

$$(\frac{x}{2})^2 - (\frac{y}{3})^2 = \sec^2 t - \tan^2 t = 1.$$

Curve is $\frac{x^2}{4} - \frac{y^2}{9} = 1$. *For* $-\pi/2 < t < \pi/2$ *we have* $\sec t > 0$. *Right half of a hyperbola.*

b. (10 points) *Find the vector equation* $\mathbf{r}(s) = \mathbf{r}_0 + s\mathbf{v}$ *of the tangent line to this curve at* $t = \frac{\pi}{4}$.

Solution: $\mathbf{R}'(t) = 2\tan t \sec t\ \mathbf{i} + 3\sec^2 t\ \mathbf{j}$, so $\mathbf{R}'(\pi/4) = \mathbf{v} = 2\sqrt{2}\ \mathbf{i} + 6\ \mathbf{j}$. $\mathbf{r}_0 = \mathbf{R}(\pi/4) = 2\sqrt{2}\ \mathbf{i} + 3\ \mathbf{j}$. Therefore

$$\mathbf{r}(s) = (2\sqrt{2}\ \mathbf{i} + 3\ \mathbf{j}) + s(2\sqrt{2}\ \mathbf{i} + 6\ \mathbf{j}).$$

Problem 2 *(20 points) Note that* $\ln 2 = \int_1^2 \frac{dx}{x}$.

a. (10 points) *Find upper and lower bounds for* $\ln 2$ *by computing the Riemann upper and lower sums for this integral, based on a regular partition* P_4 *of the interval* $[1, 2]$ *with partition width* $\Delta x = \frac{1}{4}$.

Solution: Set $f(x) = 1/x$. *partition points are* $x_j = 1 + j/4$, $j = 0, \cdots, 4$.

$$\text{upper sum} \quad U_4 = \Delta x \sum_{j=0}^{3} f(x_j) = \frac{1}{4}(\frac{4}{4} + \frac{4}{5} + \frac{4}{6} + \frac{4}{7}) = \frac{317}{420} \approx .7595$$

$$\text{lowerer sum} \quad L_4 = \Delta x \sum_{j=1}^{4} f(x_j) = \frac{1}{4}(\frac{4}{5} + \frac{4}{6} + \frac{4}{7} + \frac{4}{8}) = \frac{533}{840} \approx .6345$$

Therefore, $.7595 > \ln 2 > .6345$. *Actually,* $\ln 2 \approx .6931$.

b. (10 points) *How many partition points* n *would you need to guarantee 3 digit accuracy, i.e., what is the minimum* n *so that your upper and lower bounds for* $\ln 2$ *with partition* P_n *would differ by less than* 10^{-3}*? Is this a practical method to estimate logarithms?*

Solution: $P_n : x_j = 1 + j/n$, $j = 0, \cdots, n$, $\Delta x = 1/n$.

$$U_n = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{x_j} > \int_1^2 \frac{dx}{x} > L_n = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{x_j}.$$

Thus

$$0 < U_n - L_n = \frac{1}{n}(\frac{1}{x_0} - \frac{1}{x_n}) = \frac{1}{n}(1 - \frac{1}{2}) = \frac{1}{2n}.$$

Then $1/2n < 10^{-3} \iff 500 < n$. Smallest integer solution is $n = 501$. Not practical.

Problem 3 *(20 points) Find the arc length of the portion of the curve*

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

lying in the first quadrant. (Remark: A possible way to proceed would be to parameterize the curve in the form $x(t) = \cos^3 t,\ y(t) = \sin^3 t$. *If you use this approach, verify the parameterization.)*

Solution: In first quadrant $\iff 0 \le t \le \pi/2$.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (\cos^3 t)^{\frac{2}{3}} + (\sin^3 t)^{\frac{2}{3}} = \cos^2 t + \sin^2 t = 1.$$

$$s = \int_0^{\pi/2} ||\mathbf{R}'(t)||\ dt \quad \text{where} \quad \mathbf{R}(t) = \cos^3 t\ \mathbf{i} + \sin^3 t\ \mathbf{j}.$$

$$\mathbf{R}'(t) = -3\cos^2 t \sin t\ \mathbf{i} + 3\sin^2 t \cos t\ \mathbf{j}$$

so $||\mathbf{R}'(t)|| = 3\sin t\cos t = \frac{3}{2}\sin 2t$. *Thus* $s = 3/2$.

Problem 4 *(15 points) Identify the graphs of the given curves expressed in polar coordinates. Be as specific as you can.*

1. $r = 1/(1 - \cos\theta)$

   Solution: Conic section with $e = 1$, $p = 1$. *Parabola.*

2. $r = -5$

   Solution: Circle with radius 5, cernter at the origin.

3. $r^2 = 4r\sin\theta - 1$

   Solution: Use $r^2 = x^2 + y^2$, $r\sin\theta = y$. *Thus have* $x^2 + y^2 = 4y - 1$ *Complete the square to get* $x^2 + (y-2)^2 = 3$. *Circle with center at* $(0, 2)$ *and radius* $\sqrt{3}$.

Problem 5 *(20 points) Find the equation of the plane with normal perpendicular to the vectors* $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ *and* $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$*, and passing through the point* $P(0, 1, -1)$.

Solution: Normal to plane is $\mathbf{N} = \mathbf{A} \times \mathbf{B} = -17\ \mathbf{i} + 9\ \mathbf{j} + 7\ \mathbf{k}$. *Set* $\mathbf{R} = x\mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$*, and* $\mathbf{R}_0 = \mathbf{OP} = \ \mathbf{j} - \ \mathbf{k}$. *Equation of the plane is* $\mathbf{N} \cdot \mathbf{R} = \mathbf{N} \cdot \mathbf{R}_0 = 2$*, or* $-17x + 9y + 7z = 2$.

Problem 6 *(15 points) Find the volume of the solid generated if the region in the plane bounded by the curves* $y = x + 4$*, the* x*-axis and the* y*-axis is revolved about the line* $y = -1$.

Solution: Washer method.

$$V = \pi \int_{-4}^{0} [((x+4)+1)^2 - (1)^2]\ dx = \frac{112\pi}{3}.$$

Problem 7 *(15 points) Find the relative extrema and inflection points of the curve*

$$y = x^2 e^{-x}.$$

Sketch the curve, showing the important features.

Solution: $y' = x(2-x)\exp(-x)$, $y'' = (x^2 - 4x + 2)\exp(-x)$. *Critical points:* $x = 0, 2$. *Inflection points:* $x = 2 \pm \sqrt{2}$. *Concave up in intervals* $x < 2 - \sqrt{2}$ *and* $x > 2 + \sqrt{2}$. *Concave down in interval* $2 - \sqrt{2} < x < 2 + \sqrt{2}$. *Absolute minimum at* $x = 0$. *Relative maximum at* $x = 2$. *Line* $y = 0$ *is a horizontal asymptote as* $x \to +\infty$. $\lim_{x \to -\infty} y = +\infty$.

Problem 8 a. *(5 points) On the half-open interval* $(0, \frac{\pi}{2}]$ *find a point* c *such that the difference between the areas of the regions lying under the graphs of* $y = \sin^{50} x$ *and* $y = \cos^{50} x$ *from* 0 *to* c *has a maximal value.*

Solution: Let

$$D(z) = \int_0^z (\cos^{50} x - \sin^{50} x)\ dx, \quad 0 < z \leq \frac{\pi}{2}.$$

Find the maximum of $D(z)$ *on* $(0, \pi/2]$. *By the Fundamental Theorem of Calculus,* $dD(z)/dz = \cos^{50} z - \sin^{50} z$. *Now* $dD(z)/dz = 0 \iff \cos^{50} z = \sin^{50} z \iff \tan z = 1 \iff z = \pi/4$. *Clearly,* $dD(z)/dz > 0$ *for* $0 < z < \pi/4$ *and* $dD(z)/dz < 0$ *for* $\pi/4 < z \leq \pi/2$. *Thus there is an absolute maximum at* $z = c = \pi/4$.

b. *(5 points) Using the interval described in part [a.], find* c *such that the difference between the areas is a minimum.*

Solution: From the analysis of part a, the minimum must occur at $c = \pi/2$, *if it exists. By symmetry, it is clear that the area under the curve* $y = \cos^{50} z$ *from* $z = 0$ *to* $z = \pi/2$ *is the same as the area under the curve* $y = \cos^{50} z$ *from* $z = 0$ *to* $z = \pi/2$. *(Indeed,* $\cos(\pi/2 - z) = \sin z$.*) Thus for* $c = \pi/2$ *the difference between areas is 0, a minimum.*

Problem 9 *A full 100 gallon vat contains 100 tablespoons of sugar that is well mixed with water. Sugar water with a concentration of 5 tablespoons per gallon is flowing into the vat at a rate of 2 gallons per minute from Pipe A. Sugar water with a concentration of 10 tablespoons per gallon is flowing into the vat at a rate of 1 gallon per minute from Pipe B, and the well-mixed solution of the vat is flowing out through Pipe C at the rate of 3 gallons per minute.*

a. *(10 points) Write the differential equation and the initial conditions for this system, labeling all variables and their dimensions.*

Solution: Let $S(t)$ be the number of teblespoons of sugar in the vat at time t in minutes. Then

$$\frac{dS}{dt} = 10 + 10 - \frac{3S}{100} = 20 - \frac{3S}{100}, \quad S(0) = 100.$$

b. *(5 points) How much sugar (in tablespoons) is in the vat after one hour?*

Solution: Separate variables and use initial condition get the solution

$$S(t) = \frac{2000}{3} - \frac{1700}{3}\exp(-\frac{3t}{100}).$$

Then $S(60) \approx 572.9973$ tablespoons.

c. *(5 points) How much sugar is in the vat if the system is allowed to run indefinitely, i.e., what happens as $t \to +\infty$? Solution:*

$$\lim_{t\to+\infty} S(t) = \frac{2000}{3}.$$

Problem 10 *(15 points) A right circular cylinder open at the top is inscribed in a cone with height* 50 *and base radius* 25*. Find the dimensions of such a cylinder with the largest possible total surface area (bottom + side).*

Solution: Let h *be the height of the cylinder and* r *the radius. By similar triangles* $50/(50-h) = 25/r$ *or* $h = 50-2r$*. The surface area of the cylinder is*

$$S(r) = \pi r^2 + 2\pi r h = \pi r(100 - 3r).$$

The problem is to maximize $S(r)$ *on the domain* $0 \leq r \leq 25$*. At the endpoints* $S(0) = 0$ *and* $S(25) = 625\pi$*. We have* $S'(r) = \pi(100 - 6r)$*, so there is a critical point at* $r = 50/3$*,* S *is increasing on* $0 < r < 50/3$ *and decreasing on* $50/3 < r < 25$*. Thus the absolute maximum occurs at* $r = 50/3$*,* $h = 50/3$*.*

Problem 11 *(20 points) Use substitutions to find the following indefinite integrals. Answers without the details of the substitutions will receive no credit. Assume that* $a > 0,\ b > 0$.

a.

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}}$$

Solution: $\frac{1}{b}\arcsin(\frac{bx}{a}) + C$

b.

$$\int \frac{(\log_a x)^2}{2x}\,dx$$

Solution: $\frac{\ln a}{6}(\log_a x)^3 + C$

c.

$$\int \frac{dx}{1 + b^2(x+a)^2}$$

Solution: $\frac{1}{b}\arctan[b(x+a)] + C$

d. *Use the substitution* $\sin\theta = 3x/4$ *to evaluate*

$$\int \sqrt{16 - 9x^2}\,dx$$

Solution: $\frac{8}{3}\arcsin(\frac{3x}{4}) + \frac{x}{2}\sqrt{16 - 9x^2} + C$

Problem 12 *(10 points) The rate at which the number of bacteria in a certain colony of bacteria increases is proportional to the number of bacteria present. Exactly 10 hours after the colony is established there are 6,000 bacteria. Exactly 10 hours after that there are 8,000 bacteria. How many bacteria were used to establish the colony?*

Solution: Let $B(t)$ *be the number of bacteria at time* t *in hours. Then* $dB/dt = kB$ *which has the solution* $B(t) = B_0 \exp(kt) = B_0 K^t$ *where* $K = \exp(k)$. *We are given that*

$$B(10) = 6000 = B_0 K^{10}, \quad B(20) = 8000 = B_0 K^{20}.$$

Dividing the second equation by the first we see that $4/3 = K^{20}/K^{10} = K^{10}$. *Thus* $K = (4/3)^{1/10}$ *and* $B_0 = 6000/K^{10} = 3(6000)/4 = 4500$ *bacteria.*

Problem 13 *(20 points) Solve the initial value problem*

$$\frac{dy}{dx} = \frac{4x(y+5)}{x^2+2}, \qquad y(0) = 15.$$

When you find the solution, express y *in terms of* x.

Solution: Separate variables and integrate to get general solution $y + 5 = \exp(C)(x^2+2)^2$. *Since* $y(0) = 15$ *we have* $\exp(C) = 5$. *Thus* $y(x) = 5(x^2+2)^2 - 5$.

Problem 14 *(20 points) One end of a large tank is perpendicular to the ground and contains a circular window 2 feet in radius. The tank holds a fluid of density* ρ *lbs./ft*^{3}*. The surface of the fluid in the tank is 1 foot above the top of the window. Find the total force exerted by the fluid on the window.*

Solution: The perimeter of the window has the equation $x^2 + y^2 = 4$*. The total force* F *is given by*

$$F = \rho \int_{-2}^{2} (2x)(3-y)dy = 2\rho \int_{-2}^{2} \sqrt{4-y^2}(3-y)dy$$

$$= 6\rho \int_{-2}^{2} \sqrt{4-y^2}dy - 2\rho \int_{-2}^{2} y\sqrt{4-y^2}dy.$$

By geometry, the first integral is $1/2$ *the area of a circle of radius* 2*. By symmetry, the second integral is* 0*. Thus*

$$F = 6\rho(\frac{\pi(2)^2}{2}) = 12\rho\pi \text{ pounds.}$$