Name: ______________________

Section: ______________________

Math 1572H. Midterm Exam I February 17, 2006

There are a total of 100 points on this exam, plus one 5-point extra credit problem that you should only work if you complete the rest of the exam. To get full credit for a problem you must show the details of your work. Answers unsupported by by an argument will get little credit.

Some formulas that you may, or may not, want to use are

$$\int u\ dv = uv - \int v\ du$$

$$\sin 2x = 2\sin x\cos x, \quad \cos 2x = \cos^2 x - \sin^2 x,$$

$$2\sin^2 x = 1 - \cos 2x, \quad 2\cos^2 x = 1 + \cos 2x,$$

$$M_x = \frac{1}{2}\int \rho y^2\ dx,\ M_y = \int \rho xy\ dx,\ M = \int \rho y\ dx, \bar{x} = \frac{M_y}{M},\ \bar{y} = \frac{M_x}{M}$$

$$T_n = \frac{\Delta x}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n), \quad \Delta x = \frac{b-a}{n}$$

$$S_n = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n), \quad \Delta x = \frac{b-a}{n}, n \text{ even}$$

$$\int \sec x\ dx = \ln|\sec x + \tan x| + c, \quad \int \csc x\ dx = \ln|\csc x - \cot x| + c,$$

Problem 1 *(20 points) To derive the Kepler equation for the time behavior of solutions to the Newton gravitational equation we have to evaluate the integral*

$$I = \int \frac{d\phi}{(1 + e\cos\phi)^2}$$

where e *is the eccentricity of the trajectory. For elliptical* $(0 < e < 1)$ *or hyperbolic* $(1 < e)$ *trajectories this is fairly difficult and requires use of the general substitution* $u = \tan(\theta/2)$*. However, for parabolic trajectories* $(e = 1)$ *it is easy and can be done using the half angle formula*

$$\cos^2\frac{\phi}{2} = \frac{1}{2}(1 + \cos\phi).$$

Verify this by evaluating

$$\int \frac{d\phi}{(1 + \cos\phi)^2}$$

showing the details of the calculation.

Solution:

$$\int \frac{d\phi}{(1+\cos\phi)^2} = \frac{1}{4}\int \frac{d\phi}{\cos^4(\phi/2)} = \frac{1}{2}\int \sec^4(\phi/2)\ d(\phi/2)$$

$$= \frac{1}{2}\int (\tan^2(\phi/2) + 1)\sec^2(\phi/2)\ d(\phi/2) = \frac{1}{2}\int (\tan^2(\phi/2) + 1)\ d\tan(\phi/2)$$

$$= \frac{1}{6}\tan^3(\phi/2) + \frac{1}{2}\tan(\phi/2) + C$$

Problem 2 *(20 points) Determine if the improper integral*

$$\int_0^1 \frac{\ln(x+1)}{x^{3/2}}\,dx$$

converges and, if so, give the steps in its evaluation.

Solution:

$$\int_0^1 \frac{\ln(x+1)}{x^{3/2}}\,dx = \lim_{b\to 0+}\int_b^1 \frac{\ln(x+1)}{x^{3/2}}\,dx.$$

Set $u=\ln(x+1), dv = x^{-3/2}dx$ and integrate by parts:

$$\int_b^1 \frac{\ln(x+1)}{x^{3/2}}\,dx = -2\frac{\ln(x+1)}{\sqrt{x}}\Big|_b^1 + 2\int_b^1 \frac{dx}{\sqrt{x}(x+1)}$$

Set $w=\sqrt{x}$:

$$= -2\ln 2 + \frac{2\ln(b+1)}{\sqrt{b}} + 4\int_{\sqrt{b}}^1 \frac{dw}{1+w^2}$$

$$= -2\ln 2 + \frac{2\ln(b+1)}{\sqrt{b}} + \pi - 4\arctan\sqrt{b}.$$

By L'Hospital's rule

$$\lim_{b\to 0+}\frac{2\ln(b+1)}{\sqrt{b}} = 4\lim_{b\to 0+}\frac{1/(b+1)}{1/\sqrt{b}} = 4\lim_{b\to 0+}\frac{\sqrt{b}}{b+1} = 0,$$

Also

$$\lim_{b\to 0+}\arctan\sqrt{b} = 0.$$

Thus

$$\lim_{b\to 0+}\int_b^1 \frac{\ln(x+1)}{x^{3/2}}\,dx = -2\ln 2 + \pi,$$

converges.

Problem 3 *Recall that a bound on the maximum error in Simpson's rule for a 4-times continuously differentiable function $f(x)$ on the interval $[a, b]$ is*

$$|\int_a^b f(x)\ dx - S_n| = |E_S| \leq \frac{K(b-a)^5}{180n^4}, \quad K = \max_{x\in[a,b]} |f^{(4)}(x)|.$$

Suppose we want to approximate

$$\int_{-1}^0 e^{x^2}\ dx, \quad \text{where} \quad \frac{d^4}{dx^4}e^{x^2} = (12 + 48x^2 + 16x^4)e^{x^2}.$$

a. (10 points) *Find a bound for the maxmum possible error $|E_S|$ of Simpson's rule for this integral, with n steps.*

Solution:

$$K = \max_{x\in[-1,0)} (12 + 48x^2 + 16x^4)e^{x^2} = (12 + 48 + 16)e = 76e$$

Thus

$$|E_S| \leq \frac{76e(1)^5}{180n^4} = \frac{76e}{180n^4} \approx 1.147718994/n^4.$$

b. (10 points) *Though it appears from the above analysis that we can make the error as small as we want simply by choosing* n *sufficiently large, in reality there are computer dependent round-off and truncation errors that grow as the number of multiplications increases, so that eventually an increase in* n *leads to a decrease in accuracy. Suppose for our computer and algorithm the round-off error grows as* $10^{-10}n$*, so that the total error is modeled by*

$$E_{\text{total}} = \frac{K(b-a)^5}{180n^4} + 10^{-10}n.$$

For the integral above, about how many steps n *should you take for maximum accuracy? What is the smallest total error that you can achieve?*

Solution: Set

$$\frac{\partial E_{\text{total}}}{\partial n} = -\frac{4K(b-a)^5}{180n^5} + 10^{-10} = 0$$

to find the critical point, which is the absolute minumum. Thus

$$n_{\min} = 10^2(\frac{4K}{180})^{1/5}(b-a) \approx 135.637,$$

so $n = 135$ *or* 136*. The minimum error is*

$$E_{\text{total}} \approx \frac{K}{180n_{\min}^4} + 10^{-10}n_{\min} \approx 1.7 \times 10^{-8}.$$

Problem 4 *The region bounded by the parabola* $y = 6x - x^2$ *and the line* $y = 2x$ *is covered by a lamina of constant density* ρ.

a. (15 points) *Find* M_x, M_y *and the mass* M *for this lamina.*

Solution: The curves intersect at $x = 0, 4$.

$$M_x = \frac{\rho}{2}\int_0^4 \left[(6x - x^2)^2 - (2x)^2\right]\, dx = \frac{896}{15}\rho$$

$$M_y = \rho\int_0^4 x[(6x - x^2) - (2x)]\, dx = \frac{64}{3}\rho.$$

$$M = \rho\int_0^4 [(6x - x^2) - (2x)]\, dx = \frac{32}{3}\rho.$$

b. (5 points) *Compute the center of mass* $(\bar{x}, \bar{y})$ *of the lamina.*

Solution:

$$\bar{x} = \frac{M_y}{M} = 2, \quad \bar{y} = \frac{M_x}{M} = \frac{28}{5}.$$

Problem 5 *(20 points) Evaluate the following indefinite integral. Show all steps.*

$$\int \frac{x^2+1}{(x^2+2x+2)(x+1)}\,dx.$$

By partial fractions,

$$\frac{x^2+1}{(x^2+2x+2)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2} = \frac{2}{x+1} + \frac{-x-3}{x^2+2x+2},$$

so

$$\int \frac{x^2+1}{(x^2+2x+2)(x+1)}\,dx = 2\int \frac{dx}{x+1} + \int \frac{(-x-3)}{x^2+2x+2}\,dx$$

Now

$$\int \frac{dx}{x+1} = \ln|x+1| + C,$$

$$\begin{aligned}\int \frac{(-x-3)}{x^2+2x+2}\,dx &= -\frac{1}{2}\int \frac{(2x+2)}{x^2+2x+2}\,dx - 2\int \frac{dx}{(x+1)^2+1} \\ &= -\frac{1}{2}\ln(x^2+2x+2) - 2\arctan(x+1) + C.\end{aligned}$$

Therefore,

$$\int \frac{x^2+1}{(x^2+2x+2)(x+1)}\,dx = \ln|x+1| - \frac{1}{2}\ln(x^2+2x+2) - 2\arctan(x+1) + C.$$

Problem 6 *(EXTRA CREDIT, 5 points) Evaluate*

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

and show each step in the evaluation

Solution:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\left[\frac{e^x-1-x}{xe^x-x}\right]=\text{(using the L'H rule)}$$

$$=\lim_{x\to 0}\left[\frac{e^x-1}{xe^x+e^x-1}\right]=\text{(using the L'H rule again)}$$

$$=\lim_{x\to 0}\left[\frac{e^x}{xe^x+2e^x}\right]=\frac{1}{2}.$$