Math 1572H.   practice Final Exam   May 7, 2007

This is a closed book three-hour examination, and no notes, laptops or communication devices are allowed, but calculators are encouraged. There are a total of 250 points on the exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. The last two pages of the exam contain results from the course that you may, or may not, want to use.

Problem	Score
1.	_____
2.	_____
3.	_____
4.	_____
5.	_____
6.	_____
7.	_____
8.	_____
9.	_____
10.	_____
11.	_____
12.	_____
13.	_____
Total:	_____

Problem 1 *(20 points) Use the substitution $x^2 = \tan\theta$ to evaluate the integral*

$$\int \frac{x}{\sqrt{x^4 + 1}}\, dx.$$

Show all steps.

Problem 2 *The region bounded by the parabola $y = x^2$ and the line $y = 5x$ is covered by a lamina of density ρ.*

a. (15 points) *Find M_x, M_y and the mass M for this lamina.*

b. (5 points) *Compute the center of mass $(\bar{x}, \bar{y})$ of the lamina.*

Problem 3 *(20 points) Consider the following finite geometric series.*

$$64 + 96 + 144 + 216 + \cdots + \frac{59049}{16}$$

a. *What is the common ratio r in this series?*

b. *If 64 is the first term in the series, what number term is $59049/16$?*

c. *What is the sum of this finite series?*

Problem 4 *(20 points) Determine if each of the following infinite series converges or diverges. In each case give a reason for your answer:*

a.
$$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

b.
$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

Problem 5 *(20 points) Find the value of each of the following limits and show your work:*

a.
$$\lim_{n \to \infty} \frac{1}{x^n + x^{-n}}$$

Here, x is a fixed posititve number.

b.
$$\lim_{n \to \infty} \frac{2^{n+1} + n}{2^n + 2n}$$

Problem 6 *(20 points) Find all points (x, y) of intersection of the two curves expressed in polar coordinates by*

$$r^2 = 4\cos 2\theta, \quad r^2 = 4\sin 2\theta.$$

Problem 7 *(10 points) Find the equation in cylindrical coordinates:*

$$x^2 + +y^2 + z^2 - 7z = 0.$$

Problem 8 *(10 points) Find the equation in spherical coordinates:*

$$(x^2 + y^2 + z^2)^4 = (x^2 + y^2)^3.$$

Problem 9 *(20 points) The two surfaces*

$$S_1 : \ x^2 + y^2 - z^2 + 2xy - x + y = 0, \quad S_2 : \ x^2 + y^2 + z^2 = 6$$

intersect along the curve C which contains the point $P(1, 1, -2)$.

a. *Find the equation of the tangent plane to S_1 at P.*

b. *Find the vector equation of the tangent line to C at P.*

Problem 10 *(20 points) Find the speed and the tangential and normal components of the acceleration for the trajectory*

$$\mathbf{R} = \cos(4t)\ \mathbf{i} - \sin(4t)\ \mathbf{j}.$$

Problem 11 *(30 points)*

a. *Sketch the surface $x^2 - y^2 + z^2 = 4$.*

b. *Identify and name the cross sections. Based on the cross sections and the connectedness of the surface, what should be the name of this surface?*

c. *Find the tangent plane to this surface at* $(2, -3, 3)$.

d. *Find the vector equation of the line that is normal to the surface at* $(2, -3, 3)$.

Problem 12 *(20 points) A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate (including the boundary where $x^2 + y^2 = 1$) is heated so that the temperature T at any point (x, y) is $T = x^2 + 2y^2 - x$. Locate the hottest and coldest points of the plate and find the temperature at each of these points.*

Problem 13 *(20 points) The temperature T at any point with cartesian coordinates (x, y) in the plane at time t is given by the formula*

$$T(x, y, t) = xt - t^4 + 20y^2 + xy.$$

A fly proceeds along the x-axis with velocity $dx/dt = 3$. At the instant $t = 1$ the fly passes the point $(4, 0)$. From the fly's point of view, what is the rate of change of the temperature at this instant? (Hint: Regard the position coordinates (x, y) of the fly as functions of t.)

Some results that you may, or may not, want to use are

- Various identities:

$$\int u \, dv = uv - \int v \, du, \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$2 \sin^2 x = 1 - \cos 2x, \quad 2 \cos^2 x = 1 + \cos 2x$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + c, \quad \int \csc x \, dx = \ln |\csc x - \cot x| + c,$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1, \quad \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \quad |x| < 1$$

- Center of mass:

$$M_x = \frac{1}{2} \int \rho y^2 \, dx, \ M_y = \int \rho x y \, dx, \ M = \int \rho y \, dx, \bar{x} = \frac{M_y}{M}, \ \bar{y} = \frac{M_x}{M}$$

- Trapezoidal rule:

$$T_n = \frac{\Delta x}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n), \quad \Delta x = \frac{b-a}{n}$$

- Simpson's rule:

$$S_n = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n), \quad \Delta x = \frac{b-a}{n}, n \text{ even}$$

- p-series: $\sum_{n=1}^{\infty} 1/n^p$ diverges for $0 < p \le 1$ and converges for $p > 1$.

- The alternating series $S = \sum_{n=1}^{\infty}(-1)^n a_n$ converges if 1) $a_n \ge 0$ for all n, 2) $a_{n+1} \le a_n$ for all n, and 3) $\lim_{n\to\infty} a_n = 0$. The error $E_N = S - \sum_{n=0}^{N}(-1)^n a_n$ has the bound $|E_N| \le a_{N+1}$

16

- Taylor's formula with remainder.

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + R_N(x),$$

$$\text{where } R_N(x) = \int_0^x \frac{f^{(N+1)}(t)}{N!}(x-t)^N \, dt = f^{(N+1)}(\xi)\frac{x^{N+1}}{(N+1)!}$$

- The angle ψ between the radius vector and the tangent line to the curve (at the point of tangency) given in polar coordinates by $r = r(\theta)$ is $\tan\psi = r/r'$. Arc length in polar coordinates is given by $ds^2 = dr^2 + r^2 d\theta^2$. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

- If a parametric curve in the plane has position vector $\mathbf{R}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, the tangent vector is $\mathbf{R}' = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. The differential of arc length is $ds = \sqrt{dx^2 + dy^2} = ||\mathbf{R}'||$ and the unit tangent vector is $d/ds\mathbf{R} = dt/ds\mathbf{R}'(t)$. The curvature κ is defined by $d/ds\mathbf{T} = \kappa\,\mathbf{N}$ where $\mathbf{N}$ is the unit normal. Also $\kappa = d\phi/ds$ where ϕ is the angle between the positive x-axis and $\mathbf{T}$. In terms of Cartesian coordinates, $\kappa = (x'y'' - y'x'')/((x')^2 + (y')^2)^{3/2}$. The radius of curvature is $r = 1/\kappa$. We have $\mathbf{T} \cdot \mathbf{T} = \mathbf{N} \cdot \mathbf{N} = 1$, $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{T} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$, $\mathbf{N} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$.

- If $\mathbf{v} = \mathbf{R}'(t) = x'\,\mathbf{i} + y'\,\mathbf{j}$ is the velocity and $\mathbf{a} = \mathbf{v}'(t) = \mathbf{R}''(t) = x''\,\mathbf{i} + y''\,\mathbf{j}$ is the acceleration of a particle at time t in Cartesian coordinates, their representation in terms of tangential and normal components is

$$\mathbf{v} = \frac{ds}{dt}\,\mathbf{T}, \quad \mathbf{a} = \frac{d^2s}{dt^2}\,\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\,\mathbf{N}$$

where ds/dt is the speed.

- The normal vector to a surface $w(x, y, z) = 0$ at the point (x_0, y_0, z_0) on the surface is $\mathbf{N} = w_x(x_0, y_0, z_0)\,\mathbf{i} + w_y(x_0, y_0, z_0)\,\mathbf{j} + w_z(x_0, y_0, z_0)\,\mathbf{k}$. The equation of the tangent plane to the surface at (x_0, y_0, z_0) is

$$w_x(x_0, y_0, z_0)(x - x_0) + w_y(x_0, y_0, z_0)(y - y_0) + w_z(x_0, y_0, z_0)(z - z_0) = 0.$$

When the surface is expressed as $z = f(x, y)$ we write $w(x, y, z) = z - f(x, y) = 0$.

17

Brief solutions:

1.
$$\frac{1}{2}\ln\left|\sqrt{x^4+1}+x^2\right|+c$$

2a.
$$M_x=\frac{5^4\rho}{3},\quad M_y=\frac{5^4\rho}{12},\quad M=\frac{5^3\rho}{6}$$

2b.
$$(\bar{x},\bar{y})=(\frac{5}{2},10)$$

3a. $r=3/2$

3b. $n=11$

3c. 10946.68750

4a. diverges (p-series)

4b. By L'Hospital's rule
$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=0$$

so $\ln n<\frac{1}{2}\sqrt{n}$ for n sufficiently large. Thus for sufficiently large n we have
$$\frac{\ln}{n^2}<\frac{1}{2}\frac{\sqrt{n}}{n^2}=\frac{1}{2}\frac{1}{n^{3/2}}$$
Since the p-series $\sum\frac{1}{n^{3/2}}$ converges, the comparison test shows that $\sum\frac{\ln}{n^2}$ converges.

5a. Case 1: If $x>1$ then
$$\frac{1}{x^n+x^{-n}}<\frac{1}{x^n}\to 0,\ \text{ as } n\to\infty,$$

so $\lim_{n\to\infty}\frac{1}{x^n+x^{-n}}=0$.

Case 2: If $0<x<1$ then
$$\frac{1}{x^n+x^{-n}}<\frac{1}{x^{-n}}=x^n\to 0,\ \text{ as } n\to\infty,$$

18

so $\lim_{n\to\infty} \frac{1}{x^n+x^{-n}} = 0$.

Case 3: If $x = 1$ then

$$\frac{1}{x^n + x^{-n}} = \frac{1}{2},$$

so $\lim_{n\to\infty} \frac{1}{x^n+x^{-n}} = \frac{1}{2}$.

5b. Using the L'Hospital rule twice we find that the limit is 2.

6. Sketch the curves. There are three intersection points:

$$(0,0), \quad \left(\sqrt{\sqrt{2}+1}, \sqrt{\sqrt{2}-1}\right), \quad \left(-\sqrt{\sqrt{2}+1}, -\sqrt{\sqrt{2}-1}\right)$$

7.
$$r^2 + z^2 - 7z = 0$$

8.
$$\rho^2 = \sin^6 \phi$$

9a.
$$3x + 5y + 4z = 0$$

9b. Use one variable, say x, to parameterize the curve C. The vector equation of the tangent line is

$$\mathbf{R}(t) = (\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) + t(\mathbf{i} - \frac{5}{7}\,\mathbf{j} + \frac{1}{7}\,\mathbf{k})$$

or

$$x = 1 + t, \quad y = 1 - \frac{5}{7}t, \quad z = -2 + \frac{1}{7}t$$

10.
$$\mathbf{v} = -4\sin 4t\,\mathbf{i} - 4\cos 4t\,\mathbf{j} = 4\,\mathbf{T},$$

$$\mathbf{a} = -16\cos 4t\,\mathbf{i} + 16\sin 4t\,\mathbf{j} = -16\,\mathbf{N},$$

$$\mathbf{T} = -\sin 4t\,\mathbf{i} - \cos 4t\,\mathbf{j}, \quad \mathbf{N} = \cos 4t\,\mathbf{i} - \sin 4t\,\mathbf{j}, \quad \kappa = -1, \quad \frac{ds}{dt} = 4$$

11a. Graph of elliptic hyperboloid (actually circular hyperboliod) with the y axis as the asis of symmetry.

19

11b. Cross sections $z = z_0$ are hyperbolas. Cross sections $x = x_0$ are hyperbolas. Cross sections $y = y_0$ are circles.

11c.
$$2x + 3y + 3z = 4$$

11d.
$$\mathbf{R}(t) = (2\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k}) + t(4\,\mathbf{i} + 6\,\mathbf{j} + 6\,\mathbf{k})$$

or
$$x = 2 + 4t, \quad y = -3 + 6t, \quad z = 3 + 6t$$

12. Coldest point: $(1/2, 0)$, temperature $-1/4$. Hottest points $(-1/2, \pm\sqrt{3}/2)$ (on the boundary), temperature $9/4$

13
$$\frac{dT}{dt} = 3$$