Name: ______________________________

Section: ______________________________

Math 1572H. Practice Midterm Exam III

There are a total of 100 points on this exam, plus one 5-point extra credit problem that you should only work if you complete the rest of the exam. To get full credit for a problem you must show the details of your work. Answers unsupported by by an argument will get little credit. The last page of the exam contains results from the course that you may, or may not, want to use.

Problem	Score
1.	
2.	
3.	
4.	
5.	
6.	
Total:	

Problem 1 *(20 points) The equation of a curve, expressed in polar coordinates, is*

$$r = 3 - 3\sin\theta.$$

a. *Sketch the curve.*

b. *Find the point or points of intersection of this curve with the line given in Cartesian coordinates by* $y = \frac{\sqrt{3}}{3}x$.

Problem 2 *(20 points) What is the curvature of the curve* $y = e^{2x}$ *at the point* $(1/2, e)$*? Carry out the computation by determining the unit tangent and normal vectors and then computing* κ *via the basic formula*

$$\frac{d}{ds}\mathbf{T} = \kappa \mathbf{N}.$$

Problem 3 *(20 points) For parts [a.]-[c.] we have*

$$z = \ln(x^2 + y^2)$$

and for part [d.] $w(u, v)$ *satisfies the equation*

$$\sin(uw) + v^2 w + w^3 + 3 = 0.$$

Find the indicated partial derivatives.

a. z_x

b. z_y

c. z_{yx}

d. w_v

Problem 4 *(20 points) Set up, but do not evaluate, the definite integral in polar coordinates for the surface area of the solid obtained by rotating the curve* $r = e^{\pi\theta}$, $(0 \leq \theta \leq 1)$, *about the* x*-axis.*

Problem 5 *(20 points) Compute the equation of the tangent plane to the surface*

$$z = \frac{4x + 2y}{x - y}$$

at the point $(3, 1, 7)$.

Problem 6 *EXTRA CREDIT (5 points) Find the slope of the line tangent to the cardioid* $r = 2(1 - \cos\theta)$ *at the point on the curve in the first quadrant where* $\theta = \pi/4$. *(polar coordinates)*

Some results that you may, or may not, want to use are

- $$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}, \quad 2\sin^2 x = 1-\cos 2x, \quad 2\cos^2 x = 1+\cos 2x$$

- The angle ψ between the radius vector and the tangent line to the curve (at the point of tangency) given in polar coordinates by $r = r(\theta)$ is $\tan\psi = r/r'$. Arc length in polar coordinates is given by $ds^2 = dr^2 + r^2 d\theta^2$.

- If a parametric curve in the plane has position vector $\mathbf{R}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j}$, the tangent vector is $\mathbf{R}' = x'(t)\ \mathbf{i} + y'(t)\ \mathbf{j}$. The differential of arc length is $ds = \sqrt{dx^2+dy^2} = ||\mathbf{R}'||\ dt$ and the unit tangent vector is $\frac{d}{ds}\mathbf{R} = (ds/dt)^{-1}\mathbf{R}'(t)$. The curvature κ is defined by $\frac{d}{ds}\mathbf{T} = \kappa\ \mathbf{N}$ where $\mathbf{N}$ is the unit normal. Also $\kappa = \frac{d\phi}{ds}$ where ϕ is the angle between the positive x-axis and $\mathbf{T}$. In terms of Cartesian coordinates, $\kappa = (x'y'' - y'x'')/((x')^2 + (y')^2)^{3/2}$. The radius of curvature is $r = 1/\kappa$. We have $\mathbf{T}\cdot\mathbf{T} = \mathbf{N}\cdot\mathbf{N} = 1$, $\mathbf{T}\cdot\mathbf{N} = 0$ and $\mathbf{T} = \cos\phi\ \mathbf{i} + \sin\phi\ \mathbf{j}, \mathbf{N} = -\sin\phi\ \mathbf{i} + \cos\phi\ \mathbf{j}$.

- If $\mathbf{v} = \mathbf{R}'(t) = x'\ \mathbf{i} + y'\ \mathbf{j}$ is the velocity and $\mathbf{a} = \mathbf{v}'(t) = \mathbf{R}''(t) = x''\ \mathbf{i} + y''\ \mathbf{j}$ is the acceleration of a particle at time t in Cartesian coordinates, their representation in terms of tangential and normal components is

  $$\mathbf{v} = \frac{ds}{dt}\ \mathbf{T}, \quad \mathbf{a} = \frac{d^2 s}{dt^2}\ \mathbf{T} + \kappa(\frac{ds}{dt})^2\ \mathbf{N}$$

  where ds/dt is the speed.

- The normal vector to a surface $w(x,y,z) = 0$ at the point (x_0, y_0, z_0) on the surface is $\mathbf{N} = w_x(x_0,y_0,z_0)\ \mathbf{i} + w_y(x_0,y_0,z_0)\ \mathbf{j} + w_z(x_0,y_0,z_0)\ \mathbf{k}$. The equation of the tangent plane to the surface at (x_0,y_0,z_0) is

  $$w_x(x_0,y_0,z_0)(x-x_0) + w_y(x_0,y_0,z_0)(y-y_0) + w_z(x_0,y_0,z_0)(z-z_0) = 0.$$

  When the surface is expressed as $z = f(x,y)$ we write $w(x,y,z) = z - f(x,y) = 0$.

Brief solutions:

1a. A cardioid in vertical position.

1b. The 3 points $(0,0), (3\sqrt{3}/4, 3/4), (-9\sqrt{3}/4, -9/4)$ in Cartesian coordinates, or $[0,0], [3/2, \pi/6], [9/2, 7\pi/6]$ in polar coordinates.

2 $\kappa = \frac{e}{(1+4e^2)^{3/2}}$

3a. $\frac{2x}{x^2+y^2}$

3b. $\frac{2y}{x^2+y^2}$

3c. $-\frac{4xy}{(x^2+y^2)^2}$

3d. $-\frac{2vw}{u\cos(uw)+v^2+3w^2}$

4

$$2\pi\sqrt{1+\pi^2}\int_0^1 e^{2\pi\theta}\sin\theta \; d\theta$$

5 $3x - 9y + 2z = 7$

6 $\frac{\sqrt{2}}{2-\sqrt{2}}$