0.1 Pythagorean triples

A *Pythagorean triple* (PT) is an ordered triplet of positive integers (a, b, c) such that

$$a^2 + b^2 = c^2.$$

Clearly they correspond to the right triangles with legs of length a, b and hypotenuse c, such that a, b, c are integers. The simplest example is the familiar $(3, 4, 5)$ triangle, where $3^2 + 4^2 = 5^2$. Since the time of Pythagoras people have been interested in classifying all such triples. There are many known solutions but none are more straightforward than the solution presented here. First of all, note that for a classification the order of a and b makes no difference. Furthermore, if (a, b, c) is a Pythagorean triple then so is (ad, bd, cd) for any positive integer d. Thus the real problem reduces to finding those triples (a, b, c) such that a, b, c are relatively prime, i.e., they have no common integer factor > 1. We now limit ourselves to this case.

We can connect this problem to calculus by noting that each such PT corresponds to a rational point (x, y) on the unit circle centered at the origin. Indeed, set

$$x = \frac{a}{c}, \quad y = \frac{b}{c}.$$

If (a, b, c) is a PT then $x^2 + y^2 = (a^2 + b^2)/c^2 = 1$, so (x, y) is a rational point on the unit circle (in the first quadrant). On the other hand, suppose (x, y) is a rational point on the first quadrant of the unit circle. Then we can write $x = m_1/n_1$, $y = m_2/n_2$ where the m_i, n_i are positive integers and each pair m_1, n_1 and m_2, n_2 is relatively prime. Now we observe that $x = a/c$, $y = b/c$ where

$$a = m_1 n_2, \quad b = m_2 n_1, \quad c = n_1 n_2$$

and

$$a^2 + b^2 = m_1^2 n_2^2 + m_2^2 n_1^2 = (x^2 + y^2)c^2 = c^2.$$

Furthermore a, b, c are relatively prime. Thus the problem of finding PT's is essentially equivalent to finding the points on the unit circle with rational coordinates (x, y).

Conventionally, we parameterize the unit circle by using the rotation angle θ that the radius vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ makes with the x-axis, where θ increases in the counterclockwise direction. Thus $(x, y) = (\cos\theta, \sin\theta)$ and we sweep out the unit circle as θ varies in the domain $-\pi < \theta \leq \pi$. Thus the PT problem is the same as finding the angles θ such that $\cos\theta,\ \sin\theta$ are both rational numbers.

A solution of the problem can be obtained from a different parameterization of the unit circle, namely

$$x(t) = \frac{1-t^2}{1+t^2}, \quad y(t) = \frac{2t}{1+t^2}, \tag{1}$$

see problem 12 on page 591 of your text.

Problem 1 *Verify that for each real number t the corresponding point (x, y) lies on the unit circle.*

Problem 2 *Verify that t and θ can be related by the formula*

$$t = \tan\frac{\theta}{2}.$$

(Note: You can either verify this geometrically, as on page 591 of your text, or by using half-angle formulas.) Conclude that each point (x, y) on the unit circle is the image of a unique t, except the point $(-1, 0)$.

Problem 3 *What is the range of values of t such that $(x(t), y(t))$ covers every point in the first quadrant of the unit circle, i.e., every point such that $x(t) \geq 0,\ y(t) \geq 0$?*

It is evident that for each rational $t = m/n$ the corresponding pair (x, y) is rational. For example, if $t = 1/2$ then $x = 3/5,\ y = 4/5$ and we find the PT $(3, 4, 5)$. If $t = 2/3$ then $x = 5/13,\ y = 12/13$ and we find the PT $(5, 12, 13)$. What is amazing is that the converse is true:

Problem 4 *Show that if the point (x, y) on the unit circle is rational and $(x, y) \neq (-1, 0)$ then t is rational.*

Problem 5 *Show that the Pythagorean triple (a,b,c) can be generated by the rational number $t = b/(a + c)$. Verify this by finding the parameters corresponding to the PTs $(7, 24, 25), (16, 63, 64), (20, 21, 29)$ and $(19, 180, 181)$.*

Thus the parameterization (1) basically solves the PT classification problem. There are an infinite number of relatively prime integer solutions (a, b, c) of the equation $a^2 + b^2 = c^2$ and we obtain the solutions by choosing rational values for the parameter t.

It is worth pointing out how amazing and special this parameterization happens to be. A closely related problem in number theory is to find all integer triples (a, b, c) such that

$$a^n + b^n = c^n \tag{2}$$

where n is a fixed integer > 2. The celebrated "Fermat's last Theorem" is the conjecture that (2), in distinction to $a^2 + b^2 = c^2$, has no nontrivial solutions. After stumping mathematicians for more than a century this result was proved in the last decade by Andrew Wiles.

0.2 The integration of rational trigonometric expressions

To connect PTs with calculus directly we consider the problem of evaluating integrals of the form

$$\int R(\cos\theta, \sin\theta)\ d\theta \tag{3}$$

where R is a rational function of its arguments. We have already seen, in principle, how to evaluate analytically all integrals of the form

$$\int f(t)\ dt$$

where f is a rational function. However, (3) is not obviously of this form. Again, it is amazing that the special parameterization (1), i.e.,

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \tag{4}$$

with inverse $t = \tan(\theta/2)$ comes to our rescue.

Problem 6 *Verify that*

$$d\theta = \frac{2\ dt}{1+t^2}.$$

Thus with this substitution we find

$$\int R(\cos\theta, \sin\theta)\ d\theta = 2\int R(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})\frac{dt}{1+t^2}$$

where the integrand of the transformed integral is a rational function of t. This means that, in principle, we can evaluate explicitly all integrals of the form (3). (We should note that in practice it is often easier to evaluate such integrals by another method. However, when all else fails we can always fall back on this method with the assurance that it will work.)

Problem 7 *Use the substitution (4) to evaluate the following integrals,*

1.

$$\int \frac{d\theta}{1+\cos\theta}$$

2.

$$\int \frac{d\theta}{2+\sin\theta}$$

3.

$$\int \sec\theta\ d\theta$$

0.3 Integration of the orbital equations

Recall that the differential equation for planetary orbits takes the form

$$\frac{d\theta}{dt} = \beta(1 - e\cos\theta)^2$$

where $\beta = k^{1/2}/(a(1-e^2)^{3/2} = 2\pi/(T(1-e^2)^{3/2})$, a is the semimajor axis, e is the eccentricity ($0 \le e < 1$) and T is the period. Also recall that the polar coordinates describing the orbit are $[r, \theta]$ where

$$r = \frac{ep}{1 - e\cos\theta}$$

and the directrix is given by $p = \frac{a(1-e^2)}{e}$. The origin of coordinates is at the focus closest to perihelion, so aphelion corresponds to $\theta = 0$, and perihelion to $\theta = \pm\pi$. (Frequently, astronomers adopt coordinates $[r, \tilde{\theta}]$ related to ours by $\theta = \tilde{\theta} + \pi$ so that $\tilde{\theta} = 0$ at perihelion and

$$r = \frac{ep}{1 + e\cos\tilde{\theta}}.$$

This is only a minor difference.)

For astronomers it is usually the angle θ that can be observed or computed directly, not the distance r. Thus it is critical to know the function $\theta(t)$, called the *true anomaly*, i.e., the deviation from aphelion. To find this function we need to solve the differential equation

$$\frac{d\theta}{dt} = \beta(1 - e\cos\theta)^2, \quad \theta(0) = 0 \tag{5}$$

where we have chosen the initial position to be at aphelion. (Note that here the parameter t is time, not to be confused with the parameterization of the unit circle.)

Separating variables in the orbital equation we obtain the formal solution

$$\frac{2\pi t}{T} = (1 - e^2)^{\frac{3}{2}} \int_0^\theta \frac{d\theta}{(1 - e\cos\theta)^2}. \tag{6}$$

Thus our problem reduces to evaluating the integral

$$I = \int \frac{d\theta}{(1 - e\cos\theta)^2}.$$

This is a nontrivial integral. Rather than look for special tricks we will use the fact that the integrand is a rational function of $\cos\theta$. Thus the method of the last section applies. We make the substitution

$$\cos\theta = \frac{1 - u^2}{1 + u^2}, \quad \sin\theta = \frac{2u}{1 + u^2}$$

with inverse $u = \tan(\theta/2)$. Then we find

$$I = \int \frac{d\theta}{(1 - e\cos\theta)^2} = 2\int \frac{1 + u^2}{(1 - e + (1 + e)u^2)^2} du$$

$$= \frac{2\gamma^2}{(1 + e)^2} \int \frac{1 + u^2}{(1 + \gamma u^2)^2} du$$

where $\gamma = (1+e)/(1-e)$. We can simplify the problem slightly by rewriting the integral in the form

$$I = \frac{2\gamma^2}{(1+e)^2}\int\left[\frac{\frac{1}{\gamma}}{1+\gamma u^2}+\frac{1-\frac{1}{\gamma}}{(1+\gamma u^2)^2}\right]du$$

or, with the substitution $v = \gamma^{1/2}u$,

$$I = \frac{2\gamma^{\frac{3}{2}}}{(1+e)^2}\int\left[\frac{\frac{1}{\gamma}}{1+v^2}+\frac{1-\frac{1}{\gamma}}{(1+v^2)^2}\right]dv.$$

This breaks up into two integrals, the first of which is just the integral for the arctangent. To evaluate the second integral we use the trigonometric substitution $v = \tan\phi$. Then

$$\int\frac{dv}{(1+v^2)^2} = \int\cos^2\phi\, d\phi = \frac{1}{4}\sin 2\phi + \frac{\phi}{2} + C.$$

We can evaluate this last expression because $\sin 2\phi = 2\sin\phi\cos\phi = v/(1+v^2)$ and $\phi = \arctan v$ We conclude that

$$I = 2\sqrt{\frac{1+e}{1-e}}\frac{1}{(1+e)^2(1-e)}\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}\right)+\frac{e\sin\theta}{(1-e)(1+e)(1-e\cos\theta)}+C,$$

hence

$$\frac{2\pi t}{T} = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}\right)+e\sqrt{1-e^2}\frac{\sin\theta}{1-e\cos\theta}. \qquad (7)$$

This solution is fairly complicated but quite explicit. Unfortunately, it gives $t(\theta)$ and we want $\theta(t)$, the inverse of the relation. Beginning with Kepler himself, a great deal of effort has gone into inverting this equation. (Note that the inverse $\theta(t)$ can be defined for all t through the relation $\theta(t+T) = \theta(t)+2\pi$.)

The explicit expression (7) suggests a change of dependent variable to a new angle ψ, called the *eccentric anomaly*:

$$\tan\frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\theta}{2}. \qquad (8)$$

Then using our earlier parameterization formulas (but now applied to $\tan\frac{\psi}{2}$) we find

$$\frac{\sin\theta}{1 - e\cos\theta} = \frac{1}{\sqrt{1-e^2}}\sin\psi$$

so

$$\frac{2\pi t}{T} = \psi + e\sin\psi, \tag{9}$$

an equation with much simpler form. This is *Kepler's equation.* (The term on the left, $2\pi t/T \equiv M$ is called the *mean anomaly.* It measures the angular deviation of a planet in a circular orbit and period T.) Thus our problem reduces to inverting Kepler's equation to obtain $\psi(t)$, or $\psi[M]$. Once we have this function we can easily obtain $\theta(t)$ via

$$\theta(t) = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan(\frac{\psi}{2})\right).$$

Here ψ is defined for all t via

$$\psi(t+T) = \psi(t) + 2\pi$$

or $\psi[M+2\pi] = \psi[M] + 2\pi$. There are dozens of methods for finding $\psi[M]$ from the Kepler equation

$$M = \psi + e\sin\psi, \tag{10}$$

each with its strengths and weaknesses. Note that for fixed eccentricity e, with $0 \le e < 1$, the function $M(\psi) = \psi + e\sin\psi$ is defined for all real ψ and uniquely invertible. Indeed the derivative $M'(\psi) = 1 + e\cos\psi$ is always positive, so $M(\psi)$ is monotone increasing. By plotting a graph of ψ versus M we can easily generate some tables of values of ψ corresponding to M. However, for rocket science we need many digits high accuracy. Also we may want to understand the effect of varying the eccentricity of the orbit. We consider three approaches, two numerical and one analytic.

1. Newton's Method: This method, properly applied, always works and enables the generation of results of any desired accuracy. Using our study of this method in Math 1571H we can easily set up the algorithm and establish its convergence. Suppose we are given the eccentricity e and a number M and want to find the corresponding value of ψ such

that (10) holds. To apply Newton's method we need to express this problem as the location of a root of some function $f_M(\psi)$. We set

$$f_M(\psi) = \psi + e\sin\psi - M,$$

so that the desired ψ satisfies $f_M(\psi) = 0$. When $e = 0$ the solution is $\psi = M$ so a reasonable first guess for a root is $\psi_1 = M$. Since

$$f_M'(\psi) = 1 + e\cos\psi$$

the update step in the Newton algorithm is

$$\psi_{n+1} = \psi_n - \frac{f_M(\psi_n)}{f_M'(\psi_n)} = \psi_n - \frac{\psi_n + e\sin\psi_n - M}{1 + e\cos\psi_n}, \quad n = 1, 2, \cdots.$$

From the convergence theory for the Newton algorithm, posted online for this course, we know that the algorithm will necessarily converge to the root $\psi = \Psi$, i.e., $\lim_{n\to\infty}\psi_n = \Psi$ if, for some positive $K < 1$, we have

$$\left|\frac{f_M(\psi)f_M''(\psi)}{(f_M'(\psi))^2}\right| = \left|\frac{(\psi + e\sin\psi - M)e\sin\psi}{(1 + e\cos\psi)^2}\right| < K < 1 \qquad (11)$$

for all ψ in an open interval that includes the root Ψ and the first guess ψ_1. Since the denominator of this fraction never vanishes, it is clear that such intervals always exist. In particular, if we use a graph of the curve to find an interval that includes the root and such that $|\psi + e\sin\psi - M| < (1-e)/e$ then algorithm will be guaranteed to converge if ψ_1 belongs to this interval.

2. Successive approximations: This method is similar in spirit to Newton's method and the proof of convergence is also similar. In this case we define an updating function $T(\psi)$ by

$$T(\psi) = M - e\sin\psi$$

so that Kepler's equation becomes $\psi = T(\psi)$. For given M we know that Kepler's equation has a unique solution $\hat{\psi}$ and this is the unique fixed point of T: $T(\hat{\psi}) = \hat{\psi}$. If e is 0 then this fixed point is M, so we make the initial guess $\psi_0 = M$ for the fixed point. From this we can find a sequence of approximations to $\hat{\psi}$: $\psi_0, \psi_1, \psi_2, \cdots$ via the update rule

$$\psi_{n+1} = T(\psi_n), \quad n = 0, 1, \cdots.$$

To obtain the growth rate for the iteration we compute the derivative of $T(\psi)$:

$$T'(\psi) = -e \cos \psi.$$

Since $|T'(\psi)| = e|\cos \psi| \leq e < 1$, it follows that $\hat{\psi}$ is an attractive fixed point of T. In particular, the Mean Value Theorem says that $T(\psi) - T(\phi) = T'(\xi)(\psi - \phi)$ for some ξ between ψ and ϕ. Thus $|T(\psi) - T(\phi)| \leq e|\psi - \phi|$ for all ψ, ϕ. Hence, since $\psi_{n+1} = T(\psi_n)$ and $T(\hat{\psi}) = \hat{\psi}$, we have

$$|\psi_{n+1} - \hat{\psi}| \leq e|\psi_n - \hat{\psi}| \leq \cdots \leq e^n|\psi_0 - \hat{\psi}|.$$

Since $0 \leq e < 1$ it follows that $\psi_n \to \hat{\psi}$ as $n \to \infty$. Thus the algorithm always converges to the solution. However, as compared to the Newton method the rate of convergence can be quite slow. The best that we can guarantee is that each update will multiply the maximal error by e, thus reducing it. If e is close to 1, such as for some comets, then the number of required iterations to get a desired accuracy can be very large. For Newton's method we know that once we get sufficiently close to the solution the number of digits of accuracy will double with each update.

3. Taylor series in powers of e: Newton's method yields only a numerical approximation of ψ for given values of e and M. It would be very useful to have an analytic expression for ψ as a function of e and M, so that we could easily understand the effect of varying these parameters. One approach to this problem is to employ a Taylor series expansion for ψ in powers of e. [Note that if the eccentricity of the orbit is 0 (circular orbit) then $\psi = M$.] We consider $\psi\{e\}$, i.e., ψ as a function of e, and take the Taylor series expansion

$$\psi\{e\} \sim \sum_{n=0}^{\infty} \frac{d^n}{de^n}\psi\{0\}\frac{e^n}{n!} = \psi\{0\} + \frac{d}{de}\psi\{0\}e + \frac{d^2}{de^2}\psi\{0\}\frac{e^2}{2} + \cdots$$

If e is "small" then one can expect that taking the first few terms in this expansion (called the Lagrange series) will give a good approximation of the exact solution. (In fact, for all planets except Mercury and

Pluto, the eccentricity is less than .1, for Mercury it is about .2 and for Pluto about .24.) To compute the derivatives we assume that ψ is a differential function of e and use the method of implicit differentiation of the Kepler equation (10):

$$\psi\{0\} = M,$$

$$0 = \frac{d\psi}{de} + \sin\psi + e\frac{d\psi}{de}\psi\cos\psi \longrightarrow \frac{d\psi}{de} = -\frac{\sin\psi}{1 + e\cos\psi},$$

so

$$\frac{d}{de}\psi\{0\} = -\sin M.$$

Taking the first two terms in the Taylor series we have the approximation

$$\psi \approx M - e\sin M.$$

Problem 8 *Show that the next term in the Taylor series is*

$$\frac{d^2}{de^2}\psi\{0\} = \sin 2M.$$

Up to terms of third order, the Lagrange series is

$$\psi \approx M - e\sin M + \frac{e^2}{2}\sin 2M - \frac{e^3}{8}(3\sin 3M - \sin M).$$

The validity of the Lagrange expansion can be established in more advanced analysis courses. However, the radius of convergence of the Langrange series turns out to be $|e| \approx .66$, so for satellites with orbits of eccentricity greater than this, taking more terms in the expansion will increase rather than decrease the approximation error.

0.4 Examples of elliptic orbit calculations

We will consider a problem involving the elliptic orbit of a satellite about the Earth. (This problem and its solution are adapted from the excellent book, *Orbital Mechanics for Engineering Students* by Howard D. Curtis, Elsevier, Amsterdam, 2005.) For orbits about the Sun, the point of closest approach to the Sun is called the perihelion and the most distant point is the aphelion.

For Earth orbits these points are called the perigee and apogee, respectively. For Earth, Newton's equation is $\mathbf{r}'' = k\hat{\mathbf{r}}/r^3$ with $k = 398,600$ km^3/s^2. The radius of the Earth is about 6,378 km. In our standard coordinate system with the Earth at the origin and the apse line (x-axis) passing through the perigee and apogee, the equation of the orbit is

$$r = \frac{a(1-e^2)}{1-e\cos\theta},$$

where a is the semimajor axis. Clearly $a = (r_a + r_p)/2$ (in kilometers) where r_a, r_p are the distances of the satellite to the center of the Earth at apogee and perigee, respectively. The period of the orbit is

$$T = \frac{2\pi a^{3/2}}{k^{1/2}},$$

in seconds. The time-dependent orbit equation is

$$M = \psi + e\sin\psi$$

where $M = 2\pi t/T$ and

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\psi}{2}. \qquad (12)$$

We can assume $t = 0$ at apogee. The speed $v = ||\mathbf{r}'||$ of the satellite at a point in its orbit where it is r km. from the center of the Earth is given by the equation

$$v^2 = k(\frac{2}{r} - \frac{1}{a}).$$

Problem 9 *The geocentric elliptical orbit of a satellite is 9,600 km. from the center of the Earth at perigee and 21,000 km. from the center at apogee.*

1. *Calculate the time to go from perigee at* $\theta = 180^o$ *to the true anomaly* $\theta = 300^o$.

2. *Find the position and speed of the satellite 3 hours after perigee.*

Solution of part 1. from the (time-independent) orbit equation we have

$$r_a = \frac{a(1-e^2)}{1-e}, \quad r_p = \frac{a(1-e^2)}{1+e}$$

so $r_a/r_p = (1+e)/(1-e)$. Solving for the eccentricity e we find

$$e = \frac{r_a - r_p}{r_a + r_p}.$$

Thus we can evaluate T and e from the given data to get $T \approx$ 18,834 s. and $e = 0.37255$. For $\theta = 300^o$ we solve (12) for ψ, in radians, to get $\psi = 4.8697$. If $\tilde{t}$ is the time to go from perigee to 300^o we have $t = \tilde{t} + T/2$ so

$$M = \frac{2\pi(\tilde{t} + \frac{T}{2})}{T} = \frac{2\pi\tilde{t}}{T} + \pi.$$

On the other hand the Kepler equation gives

$$M = 4.8697 + 0.37255\sin(4.8697) \approx 4.5017.$$

Thus $\tilde{t} \approx 4077$ s. (1.1132 hr).

Solution of part 2. Note that 3 hrs/ is 10,800 s. Thus we want to know the position of the satellite at time $t = T/2 + 10,800 \approx 20,217$ s. after apogee. Thus $M = 2\pi t/T \approx 8.1047$ radians. Since this is greater than 2π the satellite has gone past apogee and are retracing the orbit. Thus we can subtract 2π from M to obtain $M \approx 0.461381$ We use this value for M and Newton's method to solve the Kepler equation for ψ, and then compute θ from (12). Starting with the guess $\psi_1 = M$ we obtain accuracy of 10^{-6} after 3 steps. To five digits, $\psi \approx 0.33788$. Then from (12) we obtain $\theta \approx 13.1571^o$. The radial coordinate and the speed can easily be obtained from this.