# Project Loiterer II: Maneuvering a Mars - bound spacecraft into a stationary orbit above the equator

Math 1572H Spring 2007

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## Contents

**Abstract**

A spacecraft carrying a communications satellite (Loiterer II), funded by the start-up Mars Telecommunications Corporation (MTC), is on its way to Mars, and presently approaching the planet on a hyperbolic trajectory in the equatorial plane and in the same counterclockwise direction the planet is rotating about its axis. Based on measurements of the position of the spacecraft at two different times, each team is to determine the precise trajectory, including the eccentricity, distance $r_p$ to the planet at periapsis, speed $v_p$ at periapsis and time of arrival at periapsis. Upon arrival at periapsis the spacecraft will fire its retrorocket and begin a Hohmann maneuver to achieve a stationary circular orbit over the equator with radius $r_s$ and speed $v_s$, to be computed. The team will need to determine the proper $\Delta v$ at hyperbolic periapsis to put the spacecraft into an elliptical orbit with apoapsis $r_A = r_s$ and periapsis $r_P = r_p$. The time of arrival at the apoapsis of the elliptical orbit will also need to be determined. Next the team will need to compute the proper $\Delta v$ for firing at apoapsis to put the spacecraft into stationary orbit. Finally, since the final orbit will not be exactly stationary, and may have a very small eccentricity $\epsilon$ the team will need to do a Taylor expansion of the orbital equation and the period in terms of powers of $\epsilon$ to determine if the orbit error is sufficiently large to require firing of the engines to correct the orbit. The results should be written up as a report to the MTC Board of Directors.


# 1 Background

The Mars Telecommunications Corporation (MTC) is a start-up company with an ambitious project: To establish planet-wide cell phone coverage on Mars. The initial part of the plan is to set up 4 satellites in stationary, equally spaced, orbits over the equator of Mars, for full coverage in the equatorial regions. While acknowledging that there is presently little demand for such coverage, due to the low population $P = 000,000$ on Mars, the MTC argues that the growth potential is great. This project involves the first of these satellites and, as such, it doesn't matter where on the equator it is directly overhead.

The MTC has had some difficulty in attracting investors. Thus they have cut costs by using college science and engineering students to plan the orbital maneuvers. They pay nothing but promise a good letter of recommendation for students that plan a successful Loiterer project. Unfortunately, the first spacecraft venture, Loiterer I, went badly. The students planned a trajectory with periapsis distance $r_p$ that was less than the radius of Mars (3,396 km.) and the spacecraft crashed on the surface. Now Loiterer II has been launched. This time the Mars trajectory was carefully planned, but again a problem occurred, due to a confusion of degree and radian measure, and contact with the spacecraft was cut off while the engines were firing to establish the trajectory to Mars. Contact has again been established but no one is sure of the present exact trajectory. However, the position of the spacecraft has been observed twice as Loiterer II approaches Mars, so on this basis the trajectory can be determined and, possibly, the mission can be saved if the rockets are fired at the correct times and with the correct $\Delta v$ to reach the stationary orbit. The students that made the reported errors were not honors students (who would never make such mistakes). Willard Miller and Hazem Hamdan have assured the MTC Board that IT Honors students, particularly those in in Math 1572H, are very capable and that they can save the Loiterer II mission. Thus you have been hired as consultants to advise the MTC concerning the orbital maneuvers needed to park Loiterer II in stationary orbit.

## 2 Technical and theoretical data

Some of this information may not be needed for the mission planning. It is up to you to decide what is relevant.

length of day on Mars: 24.62 hr.

length of day on Earth: 23.95 hrs.

angle of inclination of the equator on Mars to its orbital plane: $25.19^o$

mass of Mars: $641.9 \times 10^21$ kg.

mass of Earth: $5.974 \times 10^24$ kg.

radius of Mars: 3,396 km.

radius of Earth: 6,378 km.

$k_{Earth} = 398{,}600$ km.$^3$/s.$^2$

$k_{Mars} = 42{,}828$ km.$^3$/s.$^2$

In the standard perifocal coordinate system the possible trajectories with nonzero angular momentum are

$$r = \frac{\ell^2/k}{1 + e\cos\phi}$$

where periapsis occurs at $\phi = 0$. The energy is

$$E = v^2/2 - k/r = k^2(e^2 - 1)/2\ell^2.$$

The angular momentum is $\ell = ||\mathbf{L}||$, where

$$\mathbf{L} = \mathbf{r} \times \mathbf{r}' = \ell\ \mathbf{k}$$

and $\mathbf{r}' \equiv \mathbf{v}$ and $||\mathbf{v}|| = v$. Here, $\ell = r_p v_p$ where $r_p, v_p$ are the radius and speed at periapsis. The Kepler equation for hyperbolic trajectories is

$$M_h = -F + e\ \sinh F,\ \tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\phi}{2},\ M_h = \frac{k^2(e^2-1)^{3/2}t}{\ell^3}. \tag{1}$$

In terms of the angle $\phi$ that is 0 at periapsis, the Kepler equation for elliptic orbits takes the form

$$M_e = \psi - e\sin\psi,\ \tan\frac{\psi}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2},\ M_e = \frac{2\pi t}{T}. \tag{2}$$

(Note that this has the effect of switching the sign of $e$ in the former equation for $\theta$.) Here the period is given in terms of the constants of the motion by

$$T = \frac{2\pi}{\sqrt{k}}a^{3/2} = \frac{2\pi}{\sqrt{k}}\left(\frac{r_p + r_a}{2}\right)^{3/2}$$

where $a$ is the semi major axis and $r_p, r_a$ are the radii at periapsis and apoapsis, respectively. In the special case of a circular orbit ($e = 0$) the Kepler equations reduce to

$$\frac{2\pi t}{T} = \phi.$$

# 3 Calculating the stationary orbit

We intend to put Loiterer II in a stationary orbit over the Martian equator. The first task is to determine the required period of the circular orbit, and then to determine its radius and the velocity of the spacecraft. The relationship between these quantities and the angular momentum is given by the equations.

$$r_s = \frac{\ell_s^2}{k}, \quad v_s r_s = \ell_s.$$

Thus

$$\ell_s = \sqrt{k r_s}, \quad v_s = \sqrt{\frac{k}{r_s}}.$$

$$T_s = \frac{2\pi r_s}{v_s} = \frac{2\pi r_s^{3/2}}{k^{1/2}}.$$

**Task 1** *Compute $r_s, v_s$ and $\ell_s$ for the geostationary orbit.*

# 4 Calculating the Mars trajectory

The next set of problems is to locate Loiterer II, to determine when it will reach periapsis, and to determine the radius and speed at periapsis. Presently the spacecraft is on its way to Mars, following a trajectory of the form

$$r = \frac{\ell_1^2/k}{1 + e_1 \cos\phi}$$

in the perifocal coordinate system, where $\phi$ is the angle between the periapsis position and the spacecraft (with vertex at the focus). Contact with the spacecraft has only recently been re-established and two measurements of its position have been made. The first measurement was $[r_0, \phi_0] = [889,258.49$ km., $160.9438^o]$ and the second was $[r_1, \phi_1] = [582,775.22$ km., $158.0791^o]$ where the degree measurements have to be converted back to radians (as all savvy rocket scientists know).

**Task 2** *Compute the eccentricity $e_1$ and angular momentum $\ell_1$ for this trajectory. Check that the trajectory is hyperbolic.*

The periapsis radius and speed are related to the angular momentum by

$$\ell_1 = r_p v_p,$$

so

$$\ell_1 = \sqrt{k r_p (1 + e_1)}, \quad v_p = \sqrt{\frac{k}{r_p}(1 + e_1)}.$$

**Task 3** *Compute* $r_p$ *and* $v_p$. *Check that this trajectory will not lead to a collision with Mars, i.e., a crash. The original plan, before contact with Loiterer II was lost, was that* $r_p < r_s$. *Check to see if this is still correct.*

**Task 4** *Compute the time remaining (in seconds) between the time of the second observation and arrival at periapsis.*

We want to contact Loiterer II at 23,000 seconds before periapsis in order to program it with the precise parameters for the Hohmann maneuvers to a stationary orbit. When we make radio contact we need to know exactly where it is.

**Task 5** *Use the Kepler equation for hyperbolic trajectories and Newton's method to find the location* $[r, \phi]$ *of Loiterer II at 23,000 seconds before periapsis.*

# 5 Calculating the Hohmann maneuver to an elliptical orbit from the fly-by hyperbolic trajectory

When we fire the engines we will have new angular momentum

$$\ell_2 = r_p(v_p + \Delta v_1)$$

or

$$\Delta v_1 = \frac{\ell_2 - \ell_1}{r_p}.$$

The new (elliptic) orbit must be designed to have periapsis radius $r_p$ and apogee radius $r_a = r_s$ in order for the Hohmann transfer to be feasible. Thus the new orbit equation will be

$$r = \frac{\ell_2^2/k}{1 + e_2 \cos\phi},$$

so that

$$r_p = \frac{\ell_2^2/k}{1 + e_2}, \quad r_a = r_s = \frac{\ell_2^2/k}{1 - e_2}.$$

Simple algebra gives

$$e_2 = \frac{r_s - r_p}{r_s + r_p}, \quad \ell_2^2 = \frac{2kr_s r_p}{r_s + r_p}.$$

Solving for $\Delta v_1$ we get

$$\Delta v_1 = \sqrt{\frac{k}{r_p}}\left(\sqrt{\frac{2r_s}{r_s + r_p}} - \sqrt{1 + e_1}\right).$$

**Task 6** *Compute the new constants of the motion $e_2$ and $\ell_2$. Calculate $\Delta v_1$. Is this positive or negative, i.e., should we fire or retrofire the rocket motor?*

The period of the elliptical orbit is given by

$$T_2 = \frac{2\pi}{\sqrt{k}}\left(\frac{r_s + r_p}{2}\right)^{3/2},$$

so the time to travel from periapsis to apoapsis is $T_2/2$.

**Task 7** *Compute the time between periapsis and apoapsis for this orbit.*

# 6 Calculating the Hohmann maneuver to the stationary orbit from the elliptical orbit

For the second firing which occurs at apoapsis $r_a = r_s$, we must achieve the angular momentum $\ell_s$ of the stationary orbit. We have

$$v_a r_a = v_a r_s = \ell_2 = \sqrt{\frac{2kr_s r_p}{r_s + r_p}}$$

and after firing the motor we want

$$(v_a + \Delta v_2) r_s = \ell_s \sqrt{kr_s}.$$

Thus the delta-$v$ is

$$\Delta v_2 = \sqrt{\frac{k}{r_s}}\left(1 - \sqrt{\frac{2r_p}{r_s + r_p}}\right).$$

**Task 8** *Compute $\Delta v_2$. Is this positive or negative, i.e., should we fire or retrofire the rocket motor?*

**Task 9** *The Hohmann maneuvers that you have just carried out depended upon the requirement that the periapsis radius $r_p$ was less than the radius $r_s$ of the stationary orbit. As additional rockets are sent to occupy stationary orbits about Mars, it may happen that $r_p > r_s$. How would you alter your planning and the firings of the rocket motor to successfully execute the Hohmann maneuvers to a stationary orbit? Emphasise the differences between the Hohmann maneuvers in the two cases and make a sketch of the trajectories.*

# 7 Perturbations of the geocentric stationary orbit

When we park the spacecraft in the geocentric orbit, the orbit may be (or will become) slightly elliptical over time. We need to monitor the spacecraft orbit to ensure that it remains stationary to within strict tolerance limits. Otherwise it will become necessary to fire the rocket motor again to correct the orbit. A relatively easy way to monitor the orbit is to observe the period. From the period we can compute the eccentricity $\epsilon$ of the orbit.

For our model we assume that the equatorial orbit has the form

$$r = \frac{r_s}{1 + \epsilon \cos \phi} = \frac{\ell^2 / k}{1 + \epsilon \cos \phi}$$

where $0 \leq \epsilon << 1$. That is, the eccentricity is a very small number and the angular momentum $\ell = \sqrt{r_s / k}$ is fixed. The period of the orbit is

$$T(\epsilon) = \frac{2\pi}{\sqrt{k}} \left( \frac{r_s}{1 - \epsilon^2} \right)^{3/2} .$$

We can regard the period and the radius $r(\phi, \epsilon)$ as functions of $\epsilon$. Assume that the eccentricity is so small that you can ignore all terms of order greater than $\epsilon^2$ in the Taylor series of $T$ and $r$ about $\epsilon = 0$.

**Task 10** *Work out the Taylor series expansions of $T$ and $r$ explicitly. They should be polynomials of order 2 in $\epsilon$. We will permit the present orbit only so long as the actual period differs from the desired period, the length of the Martian day, by no more than 5 minutes. Using the model and our approximation, what is the largest value of $\epsilon$ that we can tolerate?*

**Task 11** *We need to understand how the nearly stationary orbit appears from the ground for very small* $\epsilon$. *Thus, neglecting powers of* $\epsilon$ *graeter than two, we should compute the expansion*

$$\phi(t,\epsilon) = f_0(t) + f_1(t)\epsilon + f_2(t)\epsilon^2 + O(\epsilon^3), \tag{3}$$

*where*

$$f_0(t) = \frac{k^2 t}{\ell_s^3},$$

*the result for a stationary orbit. This can be done through the use of Taylor series. Recall that the Lagrange expansion for the solution* $\psi$ *of the elliptic Kepler equation* $M = \psi + \epsilon\psi$ *is*

$$\psi = M + \epsilon \sin M + \frac{\epsilon^2}{2}\sin 2M + \cdots, \tag{4}$$

*where* $M = 2\pi t/T(\epsilon)$ *and*

$$\phi(t,\epsilon) = 2\arctan\left(\sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{\psi}{2}\right). \tag{5}$$

*One can expand this last expression for* $\phi$ *in powers of* $\epsilon$ *and then substitute the Lagrange expansion for* $\psi$ *and neglect powers exceeding* $\epsilon^2$ *to get the final result. The previous group of student consutants (not honors students) carried out the expansion of (5) and obtained the result*

$$\phi = \psi + \epsilon\sin\psi - \frac{\epsilon^2}{2}\sin 2\psi + O(\epsilon^3).$$

*Verify or correct this result. Derive expansion (3), i.e., compute* $f_1(t), f_2(t)$. *Now interpret the result.*

**a.** *First assume that* $\epsilon$ *is so small that we can neglect the* $\epsilon^2$ *term in (3). How would you describe the motion as seen from the ground?*

**b.** *Now take into account the effect of the* $\epsilon^2$ *term. How would you describe the motion as seen from the ground?*

REMARK: It is easy to show that in this approximation the curve $r(\epsilon,\phi)$ derived in Task 10 is an epicycle as a function of $\phi$, i.e., it can be expanded in linear terms of the form $\cos(n\phi)$. The Cartesian coordinates $x, y$ of the orbit in the perifocal plane are

$$(x(\epsilon,\phi), y(\epsilon,\phi)) = (r(\epsilon,\phi)\cos\phi, r(\epsilon,\phi)\sin\phi).$$

That $(x(\epsilon,\phi), y(\epsilon,\phi))$ is an epicycle follows from the addition formulas

$$\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$$

$$\cos\alpha\cos\phi = \frac{1}{2}\left(\cos(\phi+\alpha)+\cos(\phi-\alpha)\right)$$

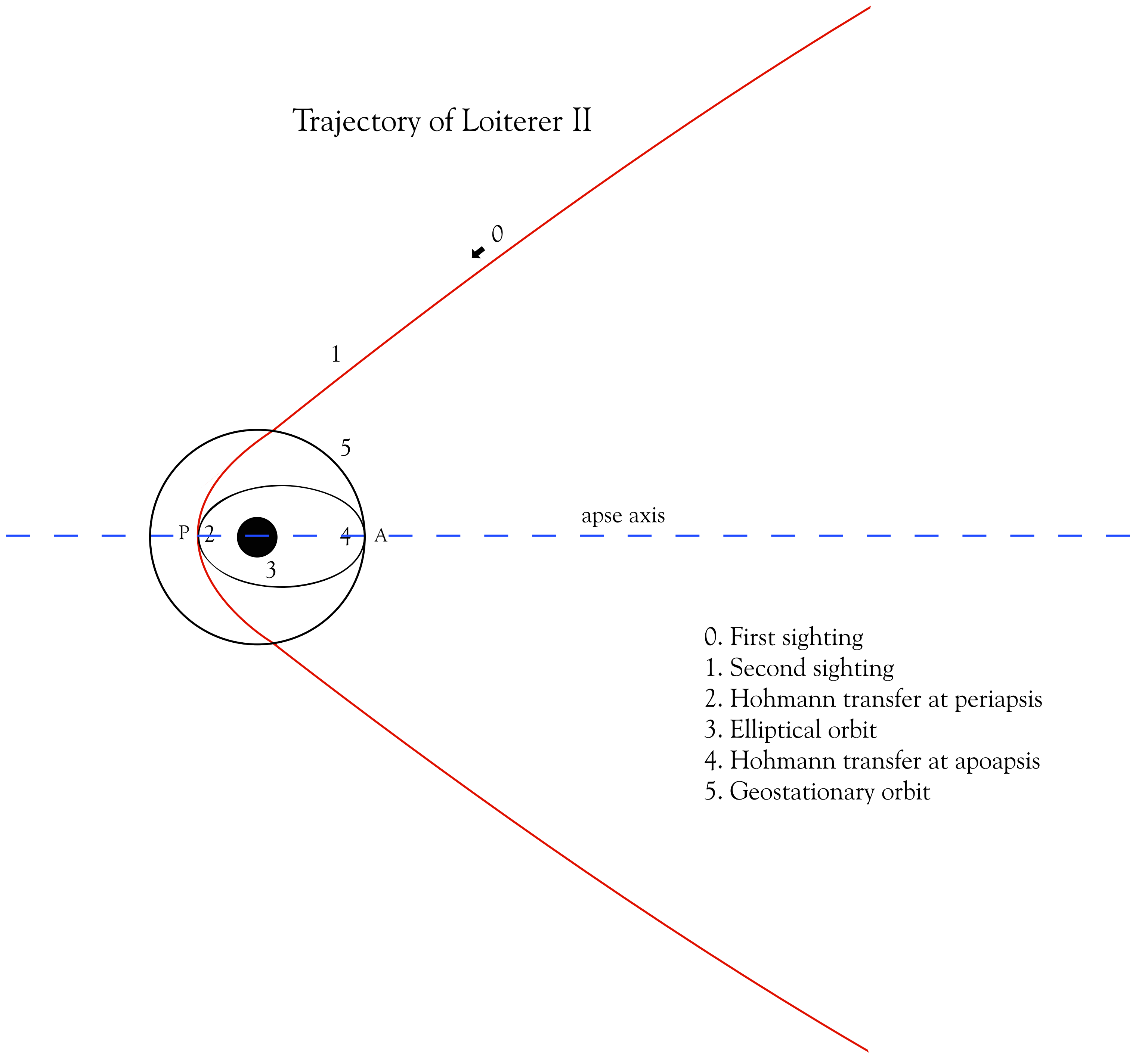
$$\cos\alpha\sin\phi = \frac{1}{2}\left(\sin(\phi+\alpha)+\sin(\phi-\alpha)\right).$$

Thus even today epicycles play some role in rocket science.

# 8 The recommendations and report

Write up your team report to the MTC, addressing each of the tasks. You can (and are encouraged to) confer with your classmates and instructors about the solutions to the problems. You can also parcel out the tasks within the team. However, each final report is the responsibility of the team members and each team will be judged as a group. All of the computations can be carried out on a calculator or using a spreadsheet program. (Indeed, standard spreadsheet programs, some of which can be downloaded for free, have all of the functions needed for the project.) The consultants report should be prepared with a word processor and should look professional. We expect complete sentences describing your findings. An initial draft will be assessed by Hazem Hamdan and then returned to you for the final report.

# 9 Appendix: The trajectory of Loiterer II


Trajectory of Loiterer II
0
1
5
apse axis
P
2
4
A
3
0. First sighting
1. Second sighting
2. Hohmann transfer at periapsis
3. Elliptical orbit
4. Hohmann transfer at apoapsis
5. Geostationary orbit