0.1 Newton's equation with gravitational force

We have seen how the formula

$$\mathbf{F} = m\mathbf{r}'' = -\frac{GmM}{r^2}\hat{\mathbf{r}} \tag{1}$$

for the motion of a planet of mass m under the gravitational action of the Sun of mass M follows from Kepler's three laws. Recall that G is the gravitational constant, the center of the sun is at the origin, and the center of the planet at time t is $\mathbf{r}(t)$ where

$$\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}, \quad r = ||\mathbf{r}|| = \sqrt{x^2 + y^2 + z^2}, \quad \hat{\mathbf{r}} = \frac{1}{r}\mathbf{r},\ ||\hat{\mathbf{r}}|| = 1.$$

Now we follow Newton and reverse the process. That is, we search for the solutions of the second order differential equation (1). We already have the solution provided with the aid of Kepler, but there are others. Factoring out the m and setting $k = GM$ we see that the differential equation becomes

$$\mathbf{r}'' = -\frac{k}{r^2}\hat{\mathbf{r}}. \tag{2}$$

To obtain a unique solution we need to specify the position and velocity of the planet at some time $t = t_0$:

$$\mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{r}'(t_0) = \mathbf{v}_0. \tag{3}$$

As you may know from your study of physics, equations (2), (3), have applicability far beyond the realm of planetary orbits. They apply to any two objects interacting through gravitational forces, the Earth and a communications satellite for example, or two molecules. They also apply to the electrical interaction of two objects with opposite charges and (with k negative) to the interaction two objects with similar charges. For simplicity, and to concentrate on the mathematical issues of calculus, I will usually assume that $M >> m$ (to be interpreted as M is much larger then m) so that the mass M object is not moved by the interaction, and it is only the motion of the mass m object that need concern us. This is the case for the Sun and the planets. However, in mechanics it is shown that, no matter what the relative sizes of M and m, the two objects orbit about their common center of mass and equation (2) still holds with $\mathbf{r}$ the vector from the center of mass to one of the objects, and M, m modified to their reduced masses. In the case $M >> m$ the center of mass can be taken as the center of the mass M object, with little loss of accuracy.

Now we turn to the solution of equations (2), (3). We have already seen through examples that finding the explicit time dependence $\mathbf{r}(t)$ of the solutions can be very complicated, whereas describing the trajectories (orbits) is much simpler. **In fact we shall see that these remarkable equations possess so much symmetry that we can determine the trajectories without integrating the equations.** The key to this will be the discovery of "constants of the motion," i.e., functions of $\mathbf{r}(t), \mathbf{r}'(t)$ that remain constant (independent of time) along a trajectory. You already know some constants of the motion, namely the energy (a scalar) and the angular momentum (a vector) that exist for many physical systems. However, the equations of gravitational interaction possess additional constants of the motion that are not shared by most other mechanical systems. The additional constants of the motion form a vector, called the Laplace vector. This vector wasn't known to Newton and it simplifies his study of the gravitational equations.

We will be looking only for solutions $\mathbf{r}(t)$ of (2) that have two continuous derivatives in the full time domain of interest, i.e, for solutions such that $\mathbf{r}(t)$ does not go to the origin.

0.2 The constants of the motion

Now we exhibit the constants of the motion that will enable us to understand the structure of equations (2). In what follows we assume that $\mathbf{r}(t)$ is a solution of (2) with initial conditions (3).

A. energy.

$$E(t) \equiv \frac{1}{2}\mathbf{r}' \cdot \mathbf{r}' - \frac{k}{r}.$$

We will show that energy is a constant of the motion for this system because $E'(t) = 0$ for all t. Indeed

$$\frac{d}{dt}E(t) = \mathbf{r}' \cdot \mathbf{r}'' + \frac{k}{r^2}r', \tag{4}$$

where we have used the Leibnitz rule for the dot product:

$$\frac{d}{dt}\mathbf{a}(t) \cdot \mathbf{b}(t) = \mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}'.$$

Now $r^2 = \mathbf{r} \cdot \mathbf{r}$ so by the Leibnitz rule

$$\frac{d}{dt} r^2 = 2\mathbf{r}' \cdot \mathbf{r}.$$

On the other hand

$$\frac{d}{dt} r^2 = 2r \frac{d}{dt} r^2 = 2rr'.$$

Thus

$$r' = \frac{1}{r} \mathbf{r}' \cdot \mathbf{r},$$

and, substituting this result and (2) into (4), we obtain

$$\frac{d}{dt} E(t) = -\frac{k}{r^3} \mathbf{r}' \cdot \mathbf{r} + \frac{k}{r^3} \mathbf{r}' \cdot \mathbf{r} = 0.$$

Thus $E(t)$ is a constant along the trajectory. In particular $E(t) = E(t_0)$, so from (3) we have

$$E(t) = \frac{1}{2} \mathbf{v}_0 \cdot \mathbf{v}_0 - \frac{k}{r_0} = \frac{1}{2} v_0^2 - \frac{k}{r_0}. \tag{5}$$

B. angular momentum.

$$\mathbf{L}(t) = \mathbf{r} \times \mathbf{r}'.$$

Angular momentum is a vector constant of the motion for this system because $\mathbf{L}'(t) = 0$ for all t. Indeed

$$\frac{d}{dt} \mathbf{L}(t) = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{r}' \times \mathbf{r}' - \frac{k}{r^3} \mathbf{r} \times \mathbf{r} = \mathbf{0}, \tag{6}$$

where we have used the Leibnitz rule for the cross product:

$$\frac{d}{dt} \mathbf{a}(t) \times \mathbf{b}(t) = \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}',$$

and the fact that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. Thus $\mathbf{L}(t)$ is a constant along the trajectory, so

$$\mathbf{L}(t) = \mathbf{r}_0 \times \mathbf{v}_0. \tag{7}$$

We can use the angular momentum to greatly simplify our problem. Indeed we can divide it into two cases: $\mathbf{L} = \mathbf{0}$ or $\mathbf{L} \neq \mathbf{0}$.

1. $\mathbf{L} = \mathbf{0}$. In this case $\mathbf{r}_0$ and $\mathbf{v}_0$ must be parallel, and as well, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ must be parallel for all t. Since the acceleration is always parallel to $\mathbf{r}(t)$ it follows that the motion occurs along a line through the origin. We can always choose our basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that the motion is along the x-axis. Thus when the angular momentum vanishes the problem becomes one-dimensional, just as discussed in section 5.5 of our text. We will say no more about this familiar case.

2. $\mathbf{L} \neq \mathbf{0}$. If the angular momentum is nonzero we see from the properties of the cross product that always $\mathbf{L} \perp \mathbf{r}(t)$ and $\mathbf{L} \perp \mathbf{r}'(t)$. This means that the solution trajectory must lie in the plane through the origin whose normal vector is $\mathbf{L}$. We can always choose our basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $\mathbf{L} = \ell\ \mathbf{k}$ where the constant $\ell \neq 0$. Thus we can always assume that the motion takes place in the $x-y$ plane. In this case we have $\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j}$ and r becomes the familiar radial coordinate. Further, introducing polar coordinates $[r, \theta]$ we have $\mathbf{r}(t) = r(t) \cos\theta(t)\ \mathbf{i} + r(t) \sin\theta(t)\ \mathbf{j}$. Note that we still have the freedom to rotate coordinates in the $x - y$ plane.

   In terms of polar coordinates we have

   $$\mathbf{r}' = r'(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}) + r\theta'(-\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j})$$

   and

   $$\hat{\mathbf{r}} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}.$$

   Thus

   $$\begin{aligned} \mathbf{L} = \mathbf{r} \times \mathbf{r}' &= \frac{r'}{r}\ \mathbf{r} \times \mathbf{r} + r\theta'\ \mathbf{r} \times (-\sin\theta\ \mathbf{i} + \cos\theta\ \mathbf{j}) \\ &= r^2\theta' \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = r^2\theta'\ \mathbf{k} = \ell\ \mathbf{k}, \end{aligned}$$

   and we conclude that

   $$\ell = r^2\theta'. \tag{8}$$

   Note that (8) implies Kepler's Second Law.

C. Laplace vector.

$$\mathbf{e}(t) = \mathbf{r}' \times \mathbf{L} - k\hat{\mathbf{r}}.$$

Again, this quantity is a vector constant of the motion for our system because $\mathbf{e}'(t) = 0$ for all t. To show this it is not necessary to take the derivative in the general case because we already know from our treatment of angular momentum that we can assume that the trajectory lies in the $x-y$ plane. Indeed with this assumtion we have

$$\mathbf{e} = \ell\ \mathbf{r}' \times \mathbf{k} - k(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}),$$

so

$$\begin{aligned}\frac{d}{dt}\mathbf{e}(t) &= \ell\mathbf{r}'' \times \mathbf{k} - k\theta'(-\sin\theta\ \mathbf{i} + \cos\theta\ bfj) \qquad (9)\\ &= -\frac{\ell k}{r^2}(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j}) \times \mathbf{k} + \frac{\ell k}{r^2}(\sin\theta\ \mathbf{i} - \cos\theta\ \mathbf{j})\\ &= -\frac{\ell k}{r^2}(\sin\theta\ \mathbf{i} - \cos\theta\ \mathbf{j}) + \frac{\ell k}{r^2}(\sin\theta\ \mathbf{i} - \cos\theta\ \mathbf{j}) = \mathbf{0}.\end{aligned}$$

Here we have made use of (8) and the fact that ℓ is a constant. Note that in these special coordinates the constant of the motion takes the form

$$\mathbf{e} = [\ell r' \sin\theta + (\ell r\theta' - k)\cos\theta]\ \mathbf{i} + [(\ell r\theta' - k)\sin\theta - \ell r' \cos\theta]\ \mathbf{j}. \qquad (10)$$

Thus this vector lies in the plane of the trajectory. Since $\mathbf{e}$ is a constant vector we can use our last bit of freedom and choose $\mathbf{i}$, $\mathbf{j}$ such that $\mathbf{e}$ is pointed in the direction of the negative x axis: $\mathbf{e} = -\mathcal{E}\ \mathbf{i}$.

0.3 The orbits of negative, zero and positive energy

Summarizing and simplifying, in the last section we showed that, in the case of nonzero angular momentum, we could choose basis coordinate vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that the trajectories lie in the $x-y$ plane and the constants of the motion expressed in terms of polar coordinates are

$$\text{angular momentum}: \quad \ell = r^2\theta', \qquad (11)$$

$$\text{energy}: \quad E = \frac{1}{2}\left((r')^2 + \frac{\ell^2}{r^2}\right) - \frac{k}{r} = \frac{1}{2}v^2 - \frac{k}{r}, \qquad (12)$$

Laplace vector :

$$-\mathcal{E} = \ell r' \sin\theta + (\frac{\ell^2}{r} - k)\cos\theta, \tag{13}$$

$$(\frac{\ell^2}{r} - k)\sin\theta - \ell r' \cos\theta = 0. \tag{14}$$

We can always assume $\ell > 0$, $\mathcal{E} \geq 0$, but E may be postitive, negative or 0.

Problem 1 *Use equations (12), (8), and (10) to show that the constants of the motion satisfy the relation*

$$||\mathbf{e}||^2 = 2||\mathbf{L}||^2 E + k^2, \text{ i.e. } k^2 e^2 = 2\ell^2 E + k^2.$$

If we solve (14) for r' and substitute the result into (13) we obtain the polar equation for the trajectories:

$$r = \frac{\frac{\ell^2}{k}}{1 - \frac{\mathcal{E}}{k}\cos\theta}. \tag{15}$$

Comparing this expression with the standard polar coordinate equation for conic sections

$$r = \frac{ep}{1 - e\cos\theta},$$

we see that the trajectories are always conic sections. In particular the eccentricity and directrix are given in terms of the constants of the motion by

$$e = \frac{\mathcal{E}}{k}, \quad p = \frac{\ell^2}{\mathcal{E}}$$

for $\mathcal{E} \neq 0$ and we have the circle $r = \ell^2/k$ with eccentricity $e = 0$ for $\mathcal{E} = 0$.

The energy E provides a simple way to distinguish between the conic sections. Note from Problem 1 that the constants of the motion are related by

$$\mathcal{E}^2 = 2\ell^2 E + k^2.$$

Case 1. $E < 0$. For negative energy we have $\mathcal{E}^2 < k^2$ or

$$0 < e = \frac{\mathcal{E}}{k} < 1.$$

This is the case of elliptical orbits.

Case 2. $E = 0$. Our identity between the constants of the motion now gives $\mathcal{E}^2 = k^2$ or

$$e = \frac{\mathcal{E}}{k} = 1.$$

This is the case of parabolic orbits. The orbit is unbounded but just a slight decrease in energy would cause it to be elliptic. An example might be a comet that approaches the Sun once but has just enough energy so that it never returns, or a space ship that has just enough energy to escape the Earth's pull, i.e., just achieve escape velocity.

Case 3. $E > 0$. For positive energy we have $\mathcal{E}^2 > k^2$ or

$$e = \frac{\mathcal{E}}{k} > 1.$$

This is the case of hyperbolic orbits. Examples are comets with more than enough energy to escape the Earth's gravitation.

Note that for all these equations the time behavior of the trajectories can be obtained from the expression $\theta' = \ell/r^2$ or

$$\theta' = \frac{k^2}{\ell^3}(1 - \frac{\mathcal{E}}{k}\cos\theta)^2. \tag{16}$$

In the following problems we will consider Earth satellites. For the Earth, $k = 398,600$ km^3/s^2, and the radius of the Earth is $R_E = 6,378$ km. The point of closest approach to the Earth of a satellite is called the perigee; the point on the orbit of greatest distance from the Earth is the apogee. Recall that the period of an elliptical orbit is related to the length of the semi-major axis via Kepler's Third Law:

$$T = \frac{2\pi a^{3/2}}{k^{1/2}}.$$

Problem 2 *A satellite is in a circular orbit, (i.e., $e = 0$, $r = \ell^2/k$) 350 km./ above the Earth's surface. Find the following*

a. *The speed in km./s. (ans. $v = 7.697$ km./s)*

b. *The period. (ans. 91 min. and 32 s.)*

Problem 3 *A satellite has an orbit of eccentricity* $e = 0.6$ *and a perigee altitude of 400 km. above the Earth surface, so that the equation for the orbit is (15) with Case 1. Find the following*

a. *The perigee radius. (ans.* $r_p = 6,778$ *km.)*

b. *The apogee radius. (ans.* $r_a = 27,110$ *km.)*

c. *The perigee velocity. (ans.* $v_p = 9,700$ *km./s)*

d. *The semimajor axis. (ans.* $a = 16,940$ *km.)*

e. *The period of the orbit (ans.* $T = 21,950$ *s.)*

Problem 4 *A spacecraft is in a geocentric trajectory. At a particular time t the spacecraft is at a distance* $r = 14,600$ *km. from the center of the Earth, at a speed* $v = 8.6$ *km./s. and at an angle* $\phi = 84.89^o$ *to perigee. Find the following*

a. *The angular momentum. (ans.* $\ell = 80,710$ *km.*^{2}*/s)*

b. *The eccentricity. (ans.* $e = 1.339$*)*

c. *The perigee radius. (ans.* $r_p = 6,986$ *km.)*

Problem 5 *A satellite is in an elliptical orbit that is 1,600 km. above the Earth surface at apogee and 600 km. above the Earth at perigee. Find the following*

a. *The eccentricity. (ans.* $e = 0.06686$*)*

b. *The speed at perigee and apogee. (ans.* $v_p = 7.81$ *km/s.,* $v_a = 10.72$ *km./s.)*

c. *The period. (ans.* $T = 107.2$ *min.)*

Problem 6 *What velocity, relative to the Earth, is required to escape the Earth on a parabolic (*$e = 1$*) path? (ans. 12.34 km./s.)*

Now we return to our initial problem, solving the Newton differential equation (2), with initial position and velocity $\mathbf{r}_0, \mathbf{v}_0$ at time t_0, (3). To solve this problem we use $\mathbf{r}_0, \mathbf{v}_0$ to compute the values of the constants of the motion $\ell, E, \mathcal{E}$. As we have seen, these values uniquely determine the trajectory of the solution. To figure out how the trajectory is traced out in time, we have to solve the first order differential equation (16) and again apply the initial conditions.