

0.1 Impulse maneuvers

Now we are ready to do some “real” rocket science and carry out orbital maneuvers. Many such operations require firing of the rocket engines for extended periods of time and get into detailed issues of physics and engineering that are not appropriate for a calculus course. However, there is one standard technique that is based almost entirely on the 7 constants of the motion for a trajectory that we have already constructed: energy (1 constant), the angular momentum vector (3 constants) and the Laplace-Runge-Lenz vector (3 constants). This is the use of impulse maneuvers to change trajectories. For an impulse maneuver the rocket engine is turned on for a very short time but with a powerful thrust. The rocket engine may have been reoriented before firing so that the thrust may not be tangent to the original trajectory. Thus at the instant t_0 of firing, the rocket which was on a trajectory with present velocity \mathbf{v}_0 and position \mathbf{r}_0 almost immediately follows a new trajectory with initial position still \mathbf{r}_0 but new velocity \mathbf{v}_1 at time t_0 . The change in velocity (or delta- v as we savvy rocket scientists like to say) is $\Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_0$, and it completely determines the new trajectory of the rocket. This is an idealization, but for very brief rocket firing during a long mission it can be quite accurate and energy efficient. The mass of the rocket will decrease due to the engine firing, but since the mass of the rocket factors out of the orbit equations, it will not have any effect on our calculations.

Now let us consider a rocket (with engines turned off) that is following a trajectory about a planet (say the Earth). Suppose at time t it is at position \mathbf{r} and has velocity $\mathbf{v} = \mathbf{r}'$. We assume that the origin of coordinates is at the center of the Earth. The constants of the motion for this trajectory are

Energy :

$$E = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} - \frac{k}{r}$$

Angular momentum :

$$\mathbf{L} = \mathbf{r} \times \mathbf{v}$$

The Laplace-Runge-Lenz vector :

$$\mathbf{e} = \mathbf{v} \times \mathbf{L} - k\hat{\mathbf{r}}.$$

We can evaluate these seven quantities at time t but they have the same values at all points on the trajectory. Although there are 7 constants, due to

the 2 conditions

$$\mathbf{e} \cdot \mathbf{L} = \mathbf{0}, \quad \|\mathbf{e}\|^2 = 2\|\mathbf{L}\|^2 E + k^2$$

there are really only 5 independent quantities. (Recall that $\|\mathbf{e}\| = ek$ and $\|\mathbf{L}\| = l$, so $e^2 k^2 = 2\ell^2 E + k^2$.) Each of these constants determines a surface in 6-dimensional position-velocity space (phase space). At any time t the rocket has coordinates (\mathbf{r}, \mathbf{v}) in phase space and, for example, the rocket coordinates must always lie on the surface $E = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} - \frac{k}{r}$ for a fixed constant E . This means, essentially, that we can use the restriction to this surface to solve for one of the 6 coordinates in terms of the other 5. Since there are 5 independent constants of the motion, the solution of Newton's equations lies on the common intersection of 5 surfaces and is a function of only 1 variable. Hence it is a 1-parameter curve. Thus these constants of the motion completely determine the trajectory without any more work. (However, they don't tell us how the trajectory is traced out in time.)

We conclude from this discussion that for an impulse maneuver at time t_0 we need merely compute the 7 constants of the motion from the values $(\mathbf{r}_0, \mathbf{v}_0)$ to determine what trajectory the rocket has been following. Then we compute new constants of the motion at t_0 from the initial values $(\mathbf{r}_0, \mathbf{v}_1)$ (where $\mathbf{v}_1 = \mathbf{v}_0 + \Delta\mathbf{v}$) to obtain the new trajectory.

0.2 An example: The Hohmann transfer

The Hohmann transfer, proposed by Hohmann in 1925, takes a satellite from a near-Earth circular orbit to a higher circular orbit with an expenditure of the minimum delta- v . To understand it we need first to determine the orbit equations for a circular orbit, i.e., an orbit with $e = 0$. This is a special case of elliptic orbits. Setting $e = 0$ in the orbit equation $r = \ell^2/k(1 - e \cos \theta)$ we see that $r = \ell^2/k$, a constant. It follows that

$$0 = \frac{d}{dt} \mathbf{r} \cdot \mathbf{r} = 2\mathbf{r} \cdot \mathbf{r}' = 2\mathbf{r} \cdot \mathbf{v},$$

so $\mathbf{r} \perp \mathbf{v}$. Thus $\ell = \|\mathbf{L}\| = rv$ and the speed v is a constant. We conclude that $v^2 = k/r$. The period of the orbit is $T_1 = 2\pi r/v = 2\pi\sqrt{r^3/k}$. The energy is $E = v^2/2 - k/r = -k/2r$.

Now suppose the the satellite is in a near-Earth circular orbit of radius r_1 and we wish to boost it to a new circular orbit of radius r_2 , ($r_2 > r_1$) and

in the same plane as the original orbit. At the time $t = t_0$ we fire the rocket motor briefly in a direction tangent to the orbit, to boost the speed from $v_1 = \sqrt{k/r_1}$ to $v_1 + \Delta v_1$. If we choose the x -axis to be the apse line through $\mathbf{r}(t_0)$ and the center of the Earth, and the $x - y$ plane as the plane of the orbit, we can assume that the position and velocity immediately after firing are $\mathbf{r}(t_0) = -r_1 \mathbf{i}$, $\mathbf{v}(t_0) = -(v_1 + \Delta v_1) \mathbf{j}$. If $\Delta v_1 > 0$, but not too large, the satellite will follow an elliptical orbit in the $x - y$ plane with equation

$$r = \frac{\ell^2/k}{1 - e \cos \theta}$$

where e is the eccentricity of the new orbit and

$$\ell = r_1(v_1 + \Delta v_1). \quad (1)$$

We want to design this orbit such that the perigee radius ($\theta = \pi$) is $r_p = r_1$ and the apogee radius ($\theta = 0$) is $r_a = r_2$, so that the ellipse will be tangent to the circular destination orbit at apogee. Thus we require

$$r_1 = \frac{\ell^2/k}{1 + e}, \quad r_2 = \frac{\ell^2/k}{1 - e}.$$

Solving for e and ℓ in these 2 equations we find

$$e = \frac{r_2 - r_1}{r_2 + r_1}, \quad \ell^2 = \frac{2kr_1r_2}{r_1 + r_2}.$$

Then, using these results and equation (1) we can solve for Δv_1 :

$$\Delta v_1 = \sqrt{\frac{k}{r_1}} \left[\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right].$$

The time for the satellite to travel from perigee to apogee is half a period:

$$T/2 = \frac{\pi}{\sqrt{k}} a^{3/2} = \frac{\pi}{\sqrt{k}} \left(\frac{r_1 + r_2}{2} \right)^{3/2}.$$

At apogee we again fire the rocket engine, briefly, to put the satellite in the higher circular orbit. As before we orient the engine so it fires in direction of the tangent vector of the trajectory. At the instant of firing, $t = t_0 + T/2$, the

speed at the apogee is v_2 and the velocity vector is tangent to the trajectory. Thus $\ell = r_1(v_1 + \Delta v_1) = r_2 v_2$, so

$$v_2 = \sqrt{\frac{2kr_1}{r_2(r_1 + r_2)}}.$$

At the instant just after firing the position is the same but the velocity is $(v_2 + \Delta v_2) \mathbf{j}$. We require that Δv_2 is exactly the change in speed required to put the satellite in the higher circular orbit, i.e., to make the eccentricity e of the orbit equal zero. Thus we require $\ell^2 = kr_2$, $e^2 k^2 = 2\ell^2 E + k^2 = 2kr_2(v_2 + \Delta v_2)^2 - k^2 = 0$ or $v_2 + \Delta v_2 = \sqrt{k/r_2}$, the speed in the higher circular orbit. The result is

$$\Delta v_2 = \sqrt{\frac{k}{r_2}} \left[1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right].$$

The total delta- v is $\Delta v_1 + \Delta v_2$ the minimum required to move to the higher circular orbit. For this new orbit we have constant speed $v_2 + \Delta v_2 = \sqrt{k/r_2}$, period $T_2 = 2\pi r_2 / (v_2 + \Delta v_2) = 2\pi \sqrt{r_2^3/k}$ and energy $E_2 = -k/2r_2$.

0.3 An example: Escaping the Earth from an elliptical orbit

Suppose we have a satellite in the elliptical orbit about the Earth

$$r = \frac{\ell_1^2/k}{1 + e_1 \cos \phi}, \quad 0 \leq e_1 < 1,$$

expressed in the usual coordinate system with ϕ the angle between perigee and the position of the satellite. We wish to fire the rocket engines, briefly, so that the satellite will escape the Earth, but use the minimum amount of fuel possible. To just barely escape the Earth we need to put the satellite in a parabolic trajectory, i.e., a trajectory with $e_2 = 1$. Since the general relation $e^2 k^2 = 2\ell^2 E + k^2$ holds this means that $E_2 = 0$ for the parabolic trajectory. The most convenient way to achieve this escape is to fire the engines at perigee in the elliptical orbit, with the change in velocity vector tangent to the orbit at perigee. The perigee distance from the Earth is

$$r_1 = \frac{\ell_1^2/k}{1 + e_1}$$

and the velocity at perigee is $\mathbf{v}_1 = -v_1 \mathbf{j}$ where $r_1 v_1 = \ell_1$. An instant after firing the engine the satellite will be at $\phi_1 = 0, r = r_1$ but with velocity $-(v_1 + \Delta v_1) \mathbf{j}$. Since the energy of the new trajectory is $E = (v_1 + \Delta v_1)^2/2 - k/r_p$, the energy will be $E = 0$ and we will just have achieved a parabolic trajectory $e_2 = 1$ provided $(v_1 + \Delta v_1)^2 = 2k/r_1$ or

$$\Delta v_1 = \sqrt{\frac{2k}{r_1}} - v_1 = \frac{k\sqrt{2(1+e_1)}}{\ell_1} \left(1 - \sqrt{\frac{1+e_1}{2}}\right).$$

The angular momentum for the parabolic trajectory is

$$\ell_2 = r_1(v_1 + \Delta v_1) = \sqrt{2kr_1} = \ell_1 \sqrt{\frac{2}{1+e_1}},$$

so the new trajectory is

$$r = \frac{2\ell_1^2}{k(1+e_1)(1+\cos\phi)}.$$

Problem 1 *A spacecraft is in circular orbit with radius r and speed v about a planet. The rocket is fired instantaneously to increase the speed in the direction of motion by $\Delta v = \alpha v$, where $\alpha > 0$. What is the eccentricity of the new orbit? (ans. $e = \alpha(\alpha + 2)$)*