Name: Solutions to problems ______________________________

Section: ______________________________

Math 2243. Lecture 020 Midterm I September 28, 2004

(There are a total of 100 points on this exam)

It is important that you show your work on each problem. Answers unsupported by details will receive little credit.

Problem 1 *Consider the differential equation*

$$\frac{dy}{dt} = t(y+2)^2.$$

a. (15 points) *Find the general solution of this equation.*

Solution:

$$\frac{dy}{(y+2)^2} = t\ dt \Longrightarrow \frac{-1}{y+2} = \frac{t^2}{2} + c$$

Solving for $y(t)$ we find

$$y(t) = \frac{-2}{t^2+2c} - 2$$

where c is an arbitrary constant. Also there is the special solution $y = -2$ that you could get from the general solution by letting $c \to \infty$.

b. (5 points) *Does the equation have an equilibrium solution $y(t) = k$ for all t? If so, find it.*

Solution: There is the equilibrium solution $y(t) = -2$. Indeed $y' = 0 = t(2-2)$.

c. (5 points) *Find the basin of attraction for this equation. That is, find the set of all initial conditions $y_0 = y(0)$ such that the solution $y(t)$ with $y(0) = y_0$ is defined for all $t \geq 0$ and approaches the equilibrium solution as $t \to +\infty$.*

Solution: In the general solution the constant c is related to $y(0)$ by $y(0) = \frac{-1}{c} - 2$, so $c = -1/(y(0)+2)$. In the general solution if y is defined at $t = 0$ then we must have $c > 0$, because otherwise y would blow up at $t = \sqrt{-2c}$. Note that $c > 0$ implies $y(0) < -2$ For $c > 0$ it is easy to see that

$$y(t) = \frac{-2}{t^2 + 2c} - 2 \to -2, \quad \text{as } t \to +\infty.$$

For the special solution $y(t) = -2$ we have $y(0) = -2$. Thus the basin of attraction is $y(0) \leq -2$.

Problem 2 a. (20 points) *Find the general solution of the linear equation*

$$y' = \frac{2y}{t} + 2t$$

using any appropriate method.

Solution:

1. **Method 1: Integrating factor method. Write the equation in the standard form** $y' - \frac{2y}{t} = 2t$**. Then** $p(t) = -2/t$ **and** $P(t) = \int \frac{-2}{t} dt = \ln(t^{-2})$**. The integrating factor is** $e^P(t) = e^{\ln(t^{-2})} = t^{-2}$**. Going back to the equation in standard form and multiplying by the integrating factor we have**

$$\frac{d}{dt}(\frac{y}{t^2}) = \frac{y'}{t^2} - 2\frac{y}{t^3} = \frac{2t}{t^2} = \frac{2}{t}.$$

   Hence,

$$\frac{d}{dt}(\frac{y}{t^2}) = \frac{2}{t} \Longrightarrow \frac{y}{t^2} = 2\ln|t| + k \Longrightarrow y(t) = 2t^2 \ln|t| + kt^2.$$

2. **Method 2: Variation of parameters. The homogeneous equation is** $y' = 2y/t$ **and separation of variables gives the solution**

$$y_H(t) = ct^2$$

   for c **an arbitrary constant. Now we assume that the solution of our original equation takes the form** $y(t) = c(t)t^2$ **for** $c(t)$ **a function. Then** $y' = c't^2 + 2ct$ **so**

$$2t = y' - \frac{2y}{t} = c't^2 + 2ct - 2ct = c't^2 \Longrightarrow c' = \frac{2}{t}.$$

   Thus, $c(t) = 2\ln|t| + k$ **and** $y(t) = c(t)t^2 = (2\ln|t| + k)t^2$**.**

b. (5 points) *Find the particular solution such that* $y(1) = 1$*.*

Solution: We have $y(t) = 2t^2 \ln|t| + kt^2$ **so if** $y(1) = 1$ **then**

$$y(1) = 1 = 2(1)^2 \ln 1 + k(1)^2 = k$$

so the particular solution is $y(t) = 2t^2 \ln|t| + t^2$**.**

Problem 3 *Consider the Euler-homogeneous equation*

$$\frac{dy}{dt} = \frac{y^2 + t^2}{yt}.$$

a. (5 points) *Apply Picard's theorem to this equation for initial condition $y(1) = 1$, i.e., find an open rectangle R, if one exists, such that $f(t, y)$ and $f_y(t, y)$ are continuous on R, and $(1, y(1)) \in R$. (Here $f(t, y) = (y^2 + t^2)/yt$.) What can you conclude?*

Solution:

$$f(t, y) = \frac{y}{t} + \frac{t}{y}, \quad f_y(t.y) = \frac{1}{t} - \frac{t}{y^2}$$

and both functions are continuous in the open rectangle

$$R = \left\{(t, y) : \frac{1}{2} < t < \frac{3}{1}, \quad \frac{1}{2} < y < \frac{3}{2}\right\}.$$

Also $(1, 1) \in R$**. Thus Picard's theorem applies and there is a unique solution** $y(t)$ **for this problem.**

b. (5 points) *Apply Picard's theorem to this equation for initial condition $y(1) = 0$.*

Solution: Both f **and** f_y **are undefined at** $(1, 0)$**, so Picard's theorem doesn't apply. Thus we are not guaranteed a solution, and if a solution exists it may not be unique.**

c. (15 points) *Solve the equation explicitly, using the fact that it is Euler-homogeneous (change of variable $u = \frac{y}{t}$) and verify your conclusions to parts a. and b.*

Solution:

$y' = u't + u$ **so the equation becomes**

$$u't + u = u + \frac{1}{u} \Longrightarrow u't = \frac{1}{u}.$$

Variables separate and we have

$$u\ du = \frac{dt}{t} \Longrightarrow \frac{u^2}{2} = \ln|t| + c.$$

Since $y = ut$ **we have**

$$y^2(t) = 2t^2 \ln|t| + 2ct^2.$$

1. part a:

$$y^2(1) = 1 = 2(1)^2 \ln|1| + 2c(1)^2 = 2c \Longrightarrow c = \frac{1}{2}.$$

Thus $y^2(t) = 2t^2 \ln|t| + t^2$ **and near** $(1,1)$ **there is the unique solution**

$$y(t) = t\sqrt{1 + 2\ln t}.$$

2. part b: Here $y(1) = 0$ **implies that** $c = 0$**. Thus** $y^2(t) = 2t^2 \ln|t|$ **and near** $(1,0)$ **there are two distinct solutions**

$$y_1(t) = t\sqrt{2\ln t}, \quad y_2(t) = -t\sqrt{2\ln t},$$

defined for $t \geq 0$**. Thus more than one solution exists.**

Problem 4 *Suppose $W(t)$ denotes the amount in milligrams of a radioactive material left after time t, in weeks. Assume that $W(0) = 10$ milligrams and that*

$$\frac{dW}{dt} = -kW(t)$$

for decay constant k.

a. (15 points) *If $W(1) = 8$ milligrams, compute k.*

Solution: The solution of the equation above is

$$W(t) = W(0)e^{-kt} = 10e^{-kt}.$$

Since $W(1) = 8$ we have $8 = 10e^{-k}$. Hence

$$\frac{4}{5} = e^{-k} \longrightarrow \ln\frac{4}{5} = \ln e^{-k} = -k \Longrightarrow k = \ln\frac{5}{4} \text{ weeks}^{-1}.$$

b. (10 points) *What is the half-life of this material?*

Solution: Let T be the half-life, in weeks. Then

$$W(T) = \frac{1}{2}W(0) = W(0)e^{-kT} \Longrightarrow \ln\frac{1}{2} = -kT,$$

so

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{\ln\frac{5}{4}} = \frac{\ln 2}{\ln 5 - \ln 4} \text{ weeks.}$$