Name: ______________________________

Section: ______________________________

Math 2243. Lecture 020   Midterm Exam II Solutions

There are a total of 100 points on this exam. The third problem is multiple choice, and you should circle the correct answer on each part. There will be no partial credit for each part of this problem. There will be partial credit awarded for the other problems but you must show your work.

Problem 1 (20 points) *Use Gauss-Jordan elimination to find the general solution of the system*

$$\begin{array}{ccccccc} x & - & y & + & z & = & 1 \\ x & + & y & & & = & -1 \\ x & + & 2y & - & z & = & 0 \end{array}$$

Solution: We write the original problem in terms of matrices as $A\mathbf{x} = \mathbf{b}$. Then we form the augmented matrix $(A : \mathbf{b})$. Performing row equivalence transforms on the augmented matrix to reduce A to RREF (details omitted) we find

$$\begin{pmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & -3 \\ 0 & 0 & 1 & : & -4 \end{pmatrix},$$

so there is a unique solution $x = 2,\ y = -3,\ z = -4$.

Problem 2

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

Compute each of the following matrices, if it exists.

a. (5 points) AB

Solution*: Not defined.*

b. (5 points) BA

Solution*:*

$$\begin{pmatrix} -1 & 1 & -2 \\ 7 & 2 & -1 \end{pmatrix}$$

c. (5 points) $2A + B$

Solution*: Not defined.*

d. (5 points) $A^{\mathrm{T}}B$

Solution*:*

$$\begin{pmatrix} 8 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Problem 3 *Initially, 50 lbs. of salt is dissolved in a tank containing 300 gal. of water. A salt solution with 2 lbs./gal. concentration is poured into the tank at 3 gal./min. The mixture, after stirring, flows from the tank at the same rate the brine is entering the tank.*

A. (10 points) *As time passes,*

1. *the amount of salt in the tank increases to an equilibrium value.*
2. *the amount of salt in the tank stays constant.*
3. *the amount of salt in the tank cycles around an equilibrium value.*
4. *the amount of salt in the tank decreases to an equilibrium value.*
5. *none of the above is observed.*

Solution: *1. The initial concentration of the salt is* $50/300 = 1/6$ *lbs./gal and the salt solution poured in has a concentration of 2 lbs./gal. Thus the concentration of salt will increase monotonically and approach the equilibium value of 2 lbs./gal.*

B. (10 points) *After 5 hours the amount of salt in the tank is, to the nearest pound,*

1. *50 lbs.*
2. *40 lbs.*
3. *600 lbs.*
4. *1000 lbs.*

Solution: 3. The amount of salt in the tank increases and approaches the equilibrium value of (2 lbs./gal)(300 gal.) $=$ 600 lbs.

Problem 4 (20 points) *Evaluate the determinant. It is important that you show the steps in your computation.*

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

Solution: Reducing the matrix to the RREF (details omitted) we see that this form is the 4×4 identity matrix and the products of the scale factors and sign switches for the row equivalence transforms is $+1$. Thus the determinant equals $+1$.

Problem 5

$$A = \begin{pmatrix} 1 & 0 & -4 \\ -1 & 1 & 7 \\ 0 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

a. (15 points) *Do the three column vectors C_1, C_2, C_3 of the matrix A form a linearly independent set or a linearly dependent set? Justify your answer.*

Solution: *We have to determine if the equation*

$$a_1C_1 + a_2C_2 + a_3C_3 = 0$$

has a solution (a_1, a_2, a_3) where not all of the a_j are zero. We can write this equation in the form $A\mathbf{a} = 0$. To solve it we put A in the RREF (details omitted):

$$\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the 3rd column is free we see that nonzero solutions exist. Indeed $a_1 = 4a_3$, $a_2 = -3a_3$. Setting $a_3 = 1$ we have in particular

$$4C_1 - 3C_2 + C_3 = 0,$$

so the set $\{C_1, C_2, C_3\}$ is linearly dependent.

b. (5 points) *Find a basis for the column space of A.*

Solution: *The RREF for A in part a. has pivots in the first 2 columns, but not the 3rd. Thus the span of $\{C_1, C_2\}$ is $\mathrm{Col}(A)$ and this set is linearly independent. Hence $\{C_1, C_2\}$ is a basis for the column space of A.*