Name: ________________________________

Section: ________________________________

Math 2243. Lecture 020 Midterm Exam III solutions

There are a total of 100 points on this exam. The first two problems are multiple choice, and you should circle the correct answers. There will be no partial credit for these problems There will be partial credit awarded for the remaining problems but you must show your work.

Problem 1 (10 points) *The solution $y(t)$ of the initial value problem*

$$y'' + 2y' + y = 0, \qquad y(0) = 6, \quad y'(0) = 0$$

1. *goes to* 0 *as* $t \to +\infty$.
2. *becomes unbounded as* $t \to +\infty$.
3. *approaches a nonzero periodic steady state as* $t \to +\infty$.
4. *remains constant for all* t.

Problem 2 (10 points) *Suppose $x(t)$ is the solution of the initial value problem*

$$x'' + 4x = 0, \qquad x(0) = 2, \quad x'(0) = 6.$$

Let $v(t) = x'(t)$. Then the phase plane trajectory $(x(t), v(t))$ of this solution

1. *lies on the constant energy curve* $\frac{1}{2}v^2 + 2x^2 = 20$.
2. *lies on the constant energy curve* $\frac{1}{2}v^2 + 4x^2 = 26$.
3. *lies on the constant energy curve* $\frac{1}{2}v^2 + 2x^2 = 26$.
4. *lies on the constant energy curve* $\frac{1}{2}v^2 + \frac{1}{2}x^2 = 20$.
5. *satisfies none of the above.*

Problem 3 *A mass-spring system with damping, and forcing term* $f(t) = \cos t$*, satisfies the equation*

$$x'' + 4x' + 5x = \cos t. \tag{1}$$

a. (8 points) *What is the transient solution of this equation?*

b. (6 points) *What is the steady state periodic solution?*

c. (6 points) *Find the solution of system (1) such that* $x(0) = x'(0) = 0$.

Problem 4

$$A = \begin{pmatrix} 1 & -3 & 2 & 4 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

a. (10 points) *Compute the kernel of A, i.e., the solution space of the homogeneous equation $A\mathbf{x} = \theta$.*

b. (5 points) *Find a basis for the kernel of A.*

c. (5 points) *Use your answer to parts a. and b., and the fact that the equation* $A\mathbf{x} = \mathbf{b}$ *has the particular solution* $\mathbf{x}_P$ *where*

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \mathbf{x}_P = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

to find the general solution $\mathbf{x}$ *of* $A\mathbf{x} = \mathbf{b}$.

Problem 5 *let* $v = P_2$ *be the vector space of polynomials in* t *of degree* ≤ 2 *and let* $T : V \to V$ *be the linear operator* $T = \frac{d}{dt}$. *The set*

$$\{1, t, t^2\}$$

is the standard basis for V.

a. (5 points) *What is the matrix of* T *with respect to this basis?*

b. (5 points) *Compute the kernel of* T *and its dimension.*

c. (5 points) *Compute the range (i.e. the image) of T and its dimension.*

d. (5 points) *What are the eigenvalues and eigenfunctions of T?*

Problem 6

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a. (10 points) *Compute the eigenvalues and eigenvectors of A.*

b. (10 points) *Solve the system $\mathbf{x}' = A\mathbf{x}$ by finding a diagonalizing matrix and decoupling.*

Solutions

1. 1

2. 3

3a.

$$x_H(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

3b.

$$x_P(t) = \frac{1}{8}(\cos t + \sin t)$$

3c.

$$x(t) = \frac{1}{8}(\cos t + \sin t)(e^{-2t} - 1)$$

4.a

$$\text{Ker}(A) = \text{ span}\left\{\begin{pmatrix} -9 \\ 7 \\ -7 \\ 11 \end{pmatrix}\right\}$$

i.e.,

$$x_1 = -\frac{9}{11}c, \quad x_2 = \frac{7}{11}c, \quad x_3 = -\frac{7}{11}c, \quad x_4 = c.$$

4.b basis

$$\left\{\begin{pmatrix} -9 \\ 7 \\ -7 \\ 11 \end{pmatrix}\right\}$$

4.c

$$\mathbf{x} = c\begin{pmatrix} -9 \\ 7 \\ -7 \\ 11 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

5.a Set

$$\mathbf{e_1} = 1, \quad \mathbf{e}_2 = t \quad \mathbf{e}_3 = t^2$$

so

$$T(\mathbf{e}_1) = T(1) = \frac{d}{dt}1 = 0, \quad T(\mathbf{e}_2) = T(t) = \frac{d}{dt}t = 1 = \mathbf{e}_1, \quad T(\mathbf{e}_3) = T(t^2) = \frac{d}{dt}t^2 = 2t = 2\mathbf{e}_2,$$

so

$$A = (T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

5.b Set $p(t) = at^2 + bt + c \in P_2$. Then $T(p)p' = 0 \longleftrightarrow a = b = 0$. Thus $p(t) = c$ and $Ker(T) = \{c\} = \text{span } \{\mathbf{e}_1\}$. dim Ker(T)= 1.

Method 2: The null space of $A\mathbf{x} = \theta$ is

$$\text{span} \left\{ (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

with dimension 1.

5.c Method 1: If $q = T(p)$ for $p \in P_2$ then q is a first order polynomial. On the other hand, any first order polynomial q can be expressed as $q = T(p) = p'$ for a second order polynomial $p \in P_2$. Hence Range(T) = span $\{\mathbf{e}_1, \mathbf{e}_2\} = \{bt+c\}$ and Rank$(T) = 2$.

Method 2:

$$\text{Range}(A) = \text{Col}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

5.d

$$\det(A - \lambda I) = -\lambda^3 = 0 \Longrightarrow \lambda = 0, 0, 0.$$

$$E_0 = \text{ span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and there is a single eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which corresponds to the eigenfunction $v(t) = 1$.

6. a. $\lambda = 0, 1, 1$

$$E_0 = \text{ span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{v}_1 \right\}$$

$$E_1 = \text{ span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{v}_2, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{v}_3 \right\}$$

6. b.

$$AP = PD, \qquad P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = P^{-1}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Set $\mathbf{y} = P\mathbf{x}$. Then the system $\mathbf{x}' = A\mathbf{x}$ becomes the decoupled system

$$y_1' = 0, \quad y_2' = y_2, \quad y_3' = y_3,$$

with general solution

$$y_1(t) = c_1, \quad y_2(t) = c_2 e^t, \quad y_3(t) = c - 3e^t.$$

Finally,

$$\mathbf{x} = P^{-1}\mathbf{y} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^t \\ c_3 e^t \end{pmatrix} = \begin{pmatrix} -c_1 \\ c_2 e^t \\ c_1 + c_3 e^t \end{pmatrix}.$$