Name: _____

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Math 2243   Practice Final Exam III

This is a closed book, closed notes exam. Basic calculators are allowed. Work all problems. The first 6 problems are multiple choice. Please circle the correct answer. There will be no partial credit for the multiple choice problems. Problems 7-16 are free response. For these problems, please do your work in the space provided and show all work. Partial credit will be given. However a correct answer may not receive full credit if the justification is incomplete or incorrect. If you need extra space, work on the back of the pages. Please clearly label all work. There are 300 points and 16 problems on the exam. The number of points for each problem is indicated.

Problem 1 (10 points) *The solution $y(t)$ of the initial value problem*

$$y'' - 3y' + 2y = 0, \qquad y(0) = 6, \quad y'(0) = 0$$

1. goes to 0 as $t \to +\infty$.

2. becomes unbounded as $t \to +\infty$.

3. approaches a periodic steady state as $t \to +\infty$.

4. remains constant for all t.

Problem 2 (10 points) *Suppose $x(t)$ is the solution of the initial value problem*

$$x'' - 2x' + 4x = 0, \qquad x(0) = 1, \quad x'(0) = 6.$$

Let $v(t) = x'(t)$. Then as t increases from 0 the phase plane trajectory $(x(t), v(t))$ of this solution

1. *lies on the constant energy curve $\frac{1}{2}v^2 + 2x^2 = 20$.*

2. *lies on the constant energy curve $\frac{1}{2}v^2 + 4x^2 = 26$.*

3. *stays within the constant energy curve $\frac{1}{2}v^2 + 2x^2 = 26$ for $t > 0$ and approaches the origin as a stable equilibrium.*

4. *stays outside of the the constant energy curve $\frac{1}{2}v^2 + 2x^2 = 20$ for $t > 0$ and is not bounded.*

5. *satisfies none of the above.*

Problem 3 (10 points) *Let $x(t)$ be the solution of the initial value problem*

$$x'' - 4x' + 4x = 3\cos 2t, \qquad x(0) = 1, \quad x'(0) = 0.$$

1. *The solution becomes unbounded as $t \to +\infty$.*

2. *The system is damped and the solution goes to zero as $t \to +\infty$.*

3. *The solution remains bounded as $t \to +\infty$.*

4. *None of the above.*

Problem 4 (10 points) *Consider the system*

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x}.$$

The equilibrium point at the origin is

   1. stable, attractive spiral.

   2. unstable, but not a saddle point or spiral.

   3. unstable, saddle point.

   4. unstable, repulsive spiral.

Problem 5 (10 points) *The 4×5 matrix A has rank 4. Then*

1. *the kernel of A has dimension 4.*

2. *the kernel of A has dimension 1.*

3. *A is invertible.*

4. *the kernel of A has dimension 0 .*

Problem 6 (10 points) *Consider the system*

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} + \mathbf{b}(t)$$

where $\mathbf{b}(t)$ *is a given nonzero function, and suppose* $\mathbf{x}_P(t)$ *is a particular solution of the system. The general solution* $\mathbf{x}(t)$ *of the system has the form*

1.
$$c_1 e^t \mathbf{v}_1 + c_2 e^t \mathbf{v}_2 + c_3 e^{2t} \mathbf{v}_3$$

2.
$$c_1 e^t \mathbf{v}_1 + c_2 e^t \mathbf{v}_2 + c_3 e^{2t} \mathbf{v}_3 + \mathbf{x}_P(t)$$

3.
$$c_1 e^t \mathbf{v}_1 + c_2 e^t (\mathbf{v}_2 + t \mathbf{v}_3) + c_3 e^{2t} \mathbf{v}_4 + \mathbf{x}_P(t)$$

4.
$$c_1 e^t \mathbf{v}_1 + c_2 t e^t \mathbf{v}_2 + c_3 e^{2t} \mathbf{v}_3 + \mathbf{x}_P(t).$$

Here, the c_j *are constants and the* $\mathbf{v}_j$ *are constant 3-tuples.*

Problem 7 (25 points) *Solve the initial value problem*

$$y'' - 4y = 2e^{-t}, \qquad y(0) = 0, \quad y'(0) = 1.$$

Problem 8 (25 points) *Is the set of polynomials*

$$\{1 - t, 1 - 2t + 2t^2, 1 - t^2\}$$

linearly dependent or independent? Justify your answer.

Problem 9 *A mass-spring system with damping, and forcing term $f(t) = \cos t$, satisfies the equation*

$$x'' + 4x' + 5x = \cos 4t. \tag{1}$$

a. (10 points) *What is the transient solution of this equation?*

b. (10 points) *What is the steady state periodic solution?*

c. (5 points) *Find the solution of system (1) such that $x(0) = x'(0) = 0$.*

Problem 10 (25 points) *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Problem 11 *Let* $V = P_2$ *be the vector space of polynomials in* t *of degree* ≤ 2 *and let* $T : V \to V$ *be the linear operator* $Tp(t) = p(2t)$, *for* $p(t) = at^2 + bt + c \in P_2$.

a. (15 points) *Find the eigenspaces of* T.

b. (5 points) *Compute the kernel of* T *and its dimension.*

c. (5 points) *Is* T *surjective? Justify your answer.*

Problem 12 (25 points) *Solve the initial value problem*

$$x'' + 2x' + x = -e^{-t}, \qquad x(0) = x'(0) = 0.$$

Problem 13 (25 points) *Find the general solution of the system*

$$\mathbf{x}' = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Problem 14 (20 points) *Use Gaussian elimination to solve the system*

$$
\begin{array}{rcrcrcl}
 & & y & + & 2z & = & 3 \\
x & + & 2y & + & z & = & 1 \\
x & + & y & & & = & 0
\end{array} \; .
$$

Verify your answer.

Problem 15 (20 points)

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Find all numbers k such that the matrix $A + kB$ has no inverse.

Problem 16 (25 points) *Find the general solution of the equation*

$$y'(t) - 2ty(t) = -t.$$

Solutions

\# 1. 2

\# 2. 4

\# 3. 1

\# 4. 3

\# 5. 2

\# 6. 3

\# 7.

$$y(t) = \frac{5}{12}e^{2t} + \frac{1}{4}e^{-2t} - \frac{2}{3}e^{-t}$$

\# 8. Linearly independent.

\# 9a.

$$x_H(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

\# 9b.

$$x_P(t) = -\frac{11}{377} \cos 4t + \frac{16}{377} \sin 4t$$

\# 9c.

$$x(t) = \frac{1}{377}\left[e^{-2t}(11 \cos t + 42 \sin t) - 11 \cos 4t + 16 \sin 4t \right]$$

\# 10. Eigenvalues $1, 1, 0, 2$.

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad E_1 = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

$$\lambda_2 = 0, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\lambda_3 = 2, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix},$$

11a. $\lambda = 4, 2, 1$

$$E_4 = \text{span}\{t^2\}, \qquad E_2 = \text{span}\{t\}, \qquad E_1 = \text{span}\{1\}$$

11b. $\text{Ker}(T) = \{0\}$, Nullity $= 0$.

11c. Surjective. Nullity + Rank $= 3 = \dim (P_2)$. Nullity $= 0$, so Rank $= 3 = \dim (P_2)$.

12.

$$x(t) = -\frac{1}{2}t^2 e^{-t}$$

13.

$$\mathbf{x}(t) = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-4t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

14. $x = 1, \quad y = -1, \quad z = 2$

15. $k = 4, \quad -1$

16.

$$y = \frac{1}{2} + ce^{t^2}$$