Name: _____

Section: _____

Math 2243. Lecture 020   Practice Midterm Exam III

There are a total of 100 points on this exam. The first two problems are multiple choice, and you should circle the correct answers. There will be no partial credit for these problems There will be partial credit awarded for the remaining problems but you must show your work.

Problem 1 (10 points) *Suppose $x(t)$ is the solution of the initial value problem*

$$x'' - 4x = 0, \qquad x(0) = 2, \quad x'(0) = 6.$$

Let $v(t) = x'(t)$. Then the phase plane trajectory $(x(t), v(t))$ of this solution

1. *lies on the curve $\frac{1}{2}v^2 + 2x^2 = 20$.*

2. *lies on the curve $\frac{1}{2}v^2 - 4x^2 = 10$.*

3. *lies on the curve $\frac{1}{2}v^2 - 2x^2 = 12$.*

4. *lies on the curve $\frac{1}{2}v^2 - \frac{1}{2}x^2 = 20$.*

5. *satisfies none of the above.*

Problem 2 (10 points) *Let $x(t)$ be the solution of the initial value problem*

$$x'' + 4x = \sin 2t, \qquad x(0) = 1, \quad x'(0) = 0.$$

1. *The system is in resonance and the solution becomes unbounded as $t \to +\infty$.*

2. *The system is damped and the solution goes to zero as $t \to +\infty$.*

3. *The solution remains bounded as $t \to +\infty$ and exhibits oscillatory behavior with beats.*

4. *None of the above.*

Problem 3 (20 points) *Solve the initial value problem*

$$y'' + 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

Problem 4

$$A = \begin{pmatrix} 1 & 6 & 2 & 4 \\ 2 & -3 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

a. (10 points) *Compute the kernel of A, i.e., the solution space of the homogeneous equation $A\mathbf{x} = \theta$, where*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad \theta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

b. (5 points) *Find a basis for the kernel of A and find its dimension.*

c. (5 points) *What is the dimension of the range of A? Justify your answer.*

Problem 5

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & -3 \\ -1 & 0 & 1 \end{pmatrix}$$

a. (10 points) *Find the eigenvalues of A.*

b. (5 points) *Find the corresponding eigenvectors.*

c. (5 points) *Can A be diagonalized by a change of basis? If so, what is the diagonal matrix?*

Problem 6 *Consider the* 2×2 *system*

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 3 & 6 \end{pmatrix} \mathbf{x}.$$

a. (15 points) *Find the general solution of this system.*

b. (5 points) *Does the system have a stable or an unstable equilibrium at the origin? Justify your answer.*

Problem

1. 5

2. 1

3.
$$y(t) = e^{-t}(1 + 3t)$$

4. a. The RREF for A is
$$\begin{pmatrix} 1 & 0 & 0 & 34/26 \\ 0 & 1 & 0 & 14/26 \\ 0 & 0 & 1 & -7/26 \end{pmatrix}$$

so
$$\text{Ker } A = \text{span}\{\mathbf{v}_1\}, \qquad \mathbf{v}_1 = \begin{pmatrix} -34 \\ -14 \\ 7 \\ 26 \end{pmatrix}$$

b. $\{\mathbf{v}_1\}$ is a basis and the dimension is 1, since the RREF has one free column.

c. The rank (i.e., the dimension of the range) is 3 since the RREF has 3 pivot columns.

5. a.
$$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1.$$

b.
$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

c.
$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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Indeed, $P^{-1}AP = D$ where

$$P = \begin{pmatrix} 3 & 0 & 0 \\ 4 & -3 & 1 \\ -3 & 2 & 0 \end{pmatrix}.$$

6. a. The eigenvalues and associated eigenvectors are

$$\lambda_1 = 2+\sqrt{19} > 0, \ \mathbf{v}_1 = \begin{pmatrix} -4+\sqrt{19} \\ 3 \end{pmatrix}, \quad \lambda_2 = 2-\sqrt{19} < 0, \ \mathbf{v}_2 = \begin{pmatrix} -4-\sqrt{19} \\ 3 \end{pmatrix},$$

and the general solution is

$$\mathbf{x}(t) = c_1 e^{(2+\sqrt{19})t}\mathbf{v}_1 + c_2 e^{(2-\sqrt{19})t}\mathbf{v}_2.$$

b. Unstable equilibrium (saddle point) at the origin, since $\lambda_1 > 0$, even though $\lambda_2 < 0$.