Nondegenerate 2D complex Euclidean superintegrable systems and algebraic varieties

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Abstract

A classical (or quantum) superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent constants of the motion polynomial in the momenta, the maximum possible. If the constants are all quadratic the system is second order superintegrable. The Kepler-Coulomb system is the best known example. Such systems have remarkable properties: multi-integrability and multi-separability, an algebra of higher order symmetries whose representation theory yields spectral information about the Schrödinger operator, deep connections with special functions and with QES systems. For complex Riemannian spaces with n=2 the structure and classification of second order superintegrable systems is complete. Here however, we present a new and conceptually simpler approach to the classification for complex Euclidean 2-space in which the possible superintegrable systems with nondegenerate potentials correspond to points on an algebraic variety. Specifically, we determine a variety in 6 variables subject to two cubic and one quartic polynomial constraints. Each point on the variety corresponds to a superintegrable system. The Euclidean group E(2,$\mathbb{C}$) acts on the variety such that two points determine the same superintegrable system if and only if they lie on the same leaf of the foliation.

1 Introduction

For any complex 2D Riemannian manifold we can always find local coordinates x, y such that the classical Hamiltonian takes the form

$$H = \frac{1}{\lambda(x,y)}(p_1^2 + p_2^2) + V(x,y), \qquad (x,y) = (x_1, x_2),$$

i.e., the complex metric is $ds^2 = \lambda(x,y)(dx^2 + dy^2)$. This system is **superintegrable** for some potential V if it admits 3 functionally independent constants of the motion (the maximum number possible) that are polynomials in the momenta p_j. It is **second order superintegrable** if the constants of the motion are quadratic, i.e., of the form $L = \sum a^{ji}(x,y)p_j p_i + W(x,y)$. There is an analogous definition of second order superintegrability for quantum systems with Schrödinger operator

$$\mathcal{H} = \frac{1}{\lambda(x,y)}(\partial_1^2 + \partial_2^2) + V(x,y)$$

and symmetry operators $\mathcal{L} = \sum \partial_j (a^{ji}(x,y))\partial_i + W(x,y)$, and these systems correspond one-to-one. Historically the most important superintegrable system is the Euclidean space Kepler-Coulomb problem where (in 2D) $V = \alpha/\sqrt{x^2+y^2}$. (Recall that this system not only has angular momentum and energy as constants of the motion but a Laplace vector that is conserved. The length of the Laplace vector can be expressed in terms of the energy and angular momentum, so that there are just 3 functionally independent constants.) As demonstrated in the literature, these systems have remarkable properties. In particular, every trajectory of a solution of the Hamilton equations for such a system in 4-dimensional phase space lies on the intersection of 3 independent constant of the motion hypersurfaces in that space, so that the trajectory can be obtained by algebraic methods alone, with no need to solve Hamilton's equations directly. Other properties include multiseparability (which implies multiintegrability, i.e., integrability in distinct ways) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], except for one isolated Euclidean system [14],) and the existence of a quadratic algebra of symmetries that closes at order 6. The quadratic algebra in the quantum case gives information relating the spectra of the constants of the motion, including the Schrödinger operator. (The existence of this quadratic algebra in the quantum Kepler-Coulomb case accounts for the fact that we can determine the energy eigenvalues for the hydrogen atom by algebraic methods alone.)

There has been recent intense activity to uncover the structure of second order superintegrable systems in n dimensions and to classify them. For the easiest case, $n = 2$, the classification is complete: [15, 16, 17, 18, 19]. (Note that we do not include the free motion case where V is a constant in the classification. The free motion case was done much earlier by Koenigs, [20].) The systems split into two classes, depending on the functional (or continuous) linear independence of the basis of functionally independent symmetries. For $n = 2$ there is only one functionally dependent superintegrable system, namely $H = 4p_z p_{\bar{z}} + V(z)$, where $V(z)$ is an arbitrary function of z alone. This system separates in only one set of coordinates $z = x + iy, \bar{z} = x - iy$. For functionally linearly independent 2D systems the theory is much more interesting. The most general potential V that defines a superintegrable system on a given Riemannian space that permits superintegrability is a solution of a system of equations of the form

$$V_{22} - V_{11} = A^{(22}(\mathbf{x})V_1 + B^{22}(\mathbf{x})V_2, \qquad V_{12} = A^{12}(\mathbf{x})V_1 + B^{12}(\mathbf{x})V_2. \tag{1}$$

If the symmetry conditions $\{H, S\} = 0$, (or $[\mathcal{H}, \mathcal{S}] = 0$ in the quantum case) provide no further conditions on the potential and if the integrability conditions for these PDEs are satisfied identically, we say that the potential is **nondegenerate**. That means, at each regular point $\mathbf{x}_0$, i.e., a (generic) point where the A^{ij}, B^{ij} are defined and analytic and the basis symmetries are linearly independent, we can prescribe the values of V, V_1, V_2 and V_{11} arbitrarily and there will exist a unique potential $V(\mathbf{x})$ with these values at $\mathbf{x}_0$. Nondegenerate potentials depend on 3 parameters, in addition to the trivial additive parameter. Degenerate potentials depend on less than 3 parameters. A basic result in 2D is that every potential V in a superintegrable system that depends on at least one multiplicative parameter is a restriction of a nondegenerate potential. Thus for a given space the classification problem reduces to finding all possible nondegenerate potentials.

In this paper we introduce a new method to carry out the classification. The main program here is to treat second order superintegrable systems by studying the coefficients C^{ij} in the equations for the potential (4) and (5). No use is made of separable coordinate systems. The systems are classified according to invariants and relative invariants [22] given by polynomials in the C^{ij}.

We consider the nondegenerate case in two-dimensional complex Euclidean space. Here it will be convenient to replace the complex Cartesian coordinates x, y by the coordinates $z, \bar{z}$. Thus for a superintegrable Hamiltonian

$$H = 4p_z p_{\bar{z}} + V \tag{2}$$

with quadratic constants of the form

$$L = a_1 p_z^2 + a_2 p_z p_{\bar{z}} + a_3 p_{\bar{z}}^2 + W \tag{3}$$

the potential is nondegenerate if

$$\frac{\partial^2 V}{\partial z^2} = C^{11}\frac{\partial V}{\partial z} + C^{12}\frac{\partial V}{\partial \bar{z}} \tag{4}$$

$$\frac{\partial^2 V}{\partial \bar{z}^2} = C^{21}\frac{\partial V}{\partial z} + C^{22}\frac{\partial V}{\partial \bar{z}} \tag{5}$$

Here V, W, a_i, C^{ij} are all functions of z and $\bar{z}$. Throughout, $z = x + iy$ and $\bar{z} = x - iy$ and all variables are allowed to take values in $\mathbb{C}$. The relation with the nondegeneracy conditions (1) is given by

$$C^{11} + C^{21} = -\frac{1}{2}(A^{22} + iB^{22}),\ C^{12} - C^{22} = -\frac{1}{2}(A^{22} - iB^{22}),$$
$$C^{11} - C^{21} = B^{12} - iA^{12},\ C^{12} - C^{22} = -B^{12} - iA^{12}.$$

Rotations have a simple action on the coefficients C^{ij}. A rotation about the origin through an angle of θ has the effect

$$z' = e^{i\theta} z\,, \quad \bar{z}' = e^{-i\theta}\bar{z} \tag{6}$$

and

$$C^{11\prime} = e^{-i\theta} C^{11}\,, \quad C^{22\prime} = e^{i\theta} C^{22}\,, \quad C^{12\prime} = e^{-3i\theta} C^{12} \quad \text{and} \quad C^{21\prime} = e^{3i\theta} C^{21} \tag{7}$$

Claim: A superintegrable system is uniquely determined by specifying the 6 numbers C^{11}, C^{22}, C^{12}, C^{21}, $C^{12}_{\bar{z}}$ and C^{21}_{z} at a regular point that satisfy three polynomial constraints, that is, a point on a three-dimensional algebraic variety in $\mathbb{C}^6$. This solution manifold is foliated into leaves on which the Euclidean group acts transitively. Each leaf corresponds to one of the nondegenerate superintegrable systems E1, E2, E3, E7, E8, E9, E10, E11, E16, E17, E19, E20 listed in [17]. (By E3 we mean an oscillator with linear terms added.) We find a quadratic algebra as a consequence of these equations and give it explicitly in terms of the C^{11}, C^{22}, C^{12}, C^{21}, $C^{12}_{\bar{z}}$, C^{21}_{z}, V_z, $V_{\bar{z}}$ and $V_{z\bar{z}}$ at a regular point. The rest of the paper is devoted to a proof of these facts.

2 Writing the system in involutive form

The condition that L is a constant of the motion, that is $\{H, L\} = 0$, gives

$$\begin{aligned}
\frac{\partial a_1}{\partial \bar{z}} &= 0\\
\frac{\partial a_3}{\partial z} &= 0\\
\frac{\partial a_1}{\partial z} + \frac{\partial a_2}{\partial \bar{z}} &= 0\\
\frac{\partial a_3}{\partial \bar{z}} + \frac{\partial a_2}{\partial z} &= 0\\
\frac{\partial W}{\partial \bar{z}} &= \frac{1}{4}\frac{\partial V}{\partial \bar{z}} a_2 + \frac{1}{2}\frac{\partial V}{\partial z} a_1\\
\frac{\partial W}{\partial z} &= \frac{1}{4}\frac{\partial V}{\partial z} a_2 + \frac{1}{2}\frac{\partial V}{\partial \bar{z}} a_3
\end{aligned}$$

The superintegrability and functional linear independence requirements mean that there are 3 linearly independent solutions of the above equations, including the Hamiltonian itself.

Equating cross partial derivatives of W gives

$$\left(2\frac{\partial a_3}{\partial \bar{z}} - \frac{\partial a_2}{\partial z} - 2a_1 C^{12} + 2a_3 C^{22}\right)\frac{\partial V}{\partial \bar{z}} - \left(2\frac{\partial a_1}{\partial z} - \frac{\partial a_2}{\partial \bar{z}} + 2a_1 C^{11} - 2a_3 C^{21}\right)\frac{\partial V}{\partial z} = 0$$

Since, by the nondegeneracy requirement, $\partial_z V$ and $\partial_{\bar z} V$ can be arbitrarily chosen at a point, their coefficients must each vanish, that is,

$$2\frac{\partial a_3}{\partial \bar z} - \frac{\partial a_2}{\partial z} - 2a_1 C^{12} + 2a_3 C^{22} = 0$$
$$2\frac{\partial a_1}{\partial z} - \frac{\partial a_2}{\partial \bar z} + 2a_1 C^{11} - 2a_3 C^{21} = 0$$

Hence all derivatives of the a_i can be expressed in terms of the a_i and C^{ij}.

Integrability conditions for the a_i yield the following expressions for derivatives of the C^{ij}.

$$\frac{\partial C^{11}}{\partial z} = \frac{2}{3}C^{22}C^{12} + \frac{2}{3}(C^{11})^2 - \frac{\partial C^{12}}{\partial \bar z} \tag{8}$$
$$\frac{\partial C^{11}}{\partial \bar z} = \frac{2}{3}C^{12}C^{21} \tag{9}$$
$$\frac{\partial C^{22}}{\partial z} = \frac{2}{3}C^{12}C^{21} \tag{10}$$
$$\frac{\partial C^{22}}{\partial \bar z} = \frac{2}{3}C^{11}C^{21} + \frac{2}{3}(C^{22})^2 - \frac{\partial C^{21}}{\partial z} \tag{11}$$
$$\frac{\partial C^{12}}{\partial z} = \frac{2}{3}C^{11}C^{12} \tag{12}$$
$$\frac{\partial C^{21}}{\partial \bar z} = \frac{2}{3}C^{22}C^{21} \tag{13}$$

If we introduce two new symbols $C^{12}_{\bar z} = \partial_{\bar z} C^{12}$ and $C^{21}_z = \partial_z C^{21}$, and use integrability conditions of (8) – (13) to solve for their derivatives, an involutive system is obtained. The additional derivatives required are

$$\frac{\partial C^{21}_z}{\partial z} = \frac{8}{9}C^{22}C^{12}C^{21} - \frac{4}{3}C^{21}C^{12}_{\bar z} + \frac{4}{9}C^{21}(C^{11})^2 + \frac{2}{3}C^{11}C^{21}_z \tag{14}$$
$$\frac{\partial C^{21}_z}{\partial \bar z} = \frac{2}{3}C^{21}_z C^{22} + \frac{4}{9}C^{12}C^{21} \tag{15}$$
$$\frac{\partial C^{12}_{\bar z}}{\partial z} = \frac{2}{3}C^{12}_{\bar z} C^{11} + \frac{4}{9}C^{12}C^{21} \tag{16}$$
$$\frac{\partial C^{12}_{\bar z}}{\partial \bar z} = \frac{8}{9}C^{11}C^{12}C^{21} - \frac{4}{3}C^{12}C^{21}_z + \frac{4}{9}C^{12}(C^{22})^2 + \frac{2}{3}C^{22}C^{12}_{\bar z} \tag{17}$$

Integrability conditions for V give a further two conditions cubic in the C^{ij}

$$9C^{21}C^{12}_{\bar z} - 3C^{11}C^{21}_z - 2C^{22}C^{12}C^{21} - 2C^{21}(C^{11})^2 = 0 \tag{18}$$
$$9C^{12}C^{21}_z - 3C^{22}C^{12}_{\bar z} - 2C^{11}C^{12}C^{21} - 2C^{12}(C^{22})^2 = 0 \tag{19}$$

and derivatives of these introduce one further independent condition, quartic in the C^{ij},

$$9C^{12}_{\bar z}C^{21}_z - 3C^{11}C^{21}C^{12}_{\bar z} - 3C^{22}C^{12}C^{21}_z + (C^{12})^2(C^{21})^2 - 2C^{11}C^{22}C^{12}C^{21} = 0\,. \tag{20}$$

So, subject to the three constraints, (18), (19) and (20), we now have an involutive system for the a_i, V, C^{ij} and W. The systems of equations (8) – (20) are invariant under rotations.

Thus we have an algebraic variety consisting of 6-tuples subject to the conditions (18), (19) and (20).

Theorem 1. *Corresponding to each 6-tuple in the algebraic variety there is one and only one Euclidean superintegrable system that agrees with this 6-tuple at some regular point.*

3 Solving the equations

Notice that equations (18) and (19) can be used to solve for $C^{12}_{\bar{z}}$ and C^{21}_{z} if and only if

$$D = 9C^{21}C^{12} - C^{11}C^{22} \neq 0\,.$$

So we consider two cases, $D = 0$ and $D \neq 0$. (Here zero or nonzero means as a function of z and $\bar{z}$, not just at a point. Indeed, using the Bertrand-Darboux equations [15] it is easy to see that the C^{ij} must be rational functions of z and $\bar{z}$. In particular they are analytic and their only singularities are poles.)

3.1 $D = 0$

Using $D = 0$, $\partial_z D = \partial_{\bar{z}} D = 0$ lead to

$$\begin{aligned} C^{12}C^{21}\left(27C^{21}_{z} - 2(C^{22})^2\right) &= 0 \\ C^{12}C^{21}\left(27C^{12}_{\bar{z}} - 2(C^{11})^2\right) &= 0\,. \end{aligned}$$

If $C^{12}C^{21} \neq 0$ we find

$$C^{12}_{\bar{z}} = \frac{2}{27}(C^{11})^2 \qquad \text{and} \qquad C^{21}_{z} = \frac{2}{27}(C^{22})^2\,.$$

Substituting these into the involutive system for the C^{ij}, C^{21}_{z} and $C^{12}_{\bar{z}}$ gives

$$\begin{aligned} C^{11}\left(7(C^{11})^2 + 9C^{22}C^{12}\right) &= 0, \\ C^{22}\left(25(C^{11})^2 + 63C^{22}C^{12}\right) &= 0, \\ C^{22}\left(7(C^{22})^2 + 9C^{11}C^{21}\right) &= 0, \\ C^{11}\left(25(C^{22})^2 + 63C^{11}C^{21}\right) &= 0, \end{aligned}$$

which are inconsistent. Hence, $C^{12}C^{21} = 0$ and since $D = 0$, we must also have $C^{11}C^{22} = 0$. So there are four cases.

[A] $C^{11} = C^{12} = 0$

[B] $C^{11} = C^{21} = 0$

[C] $C^{22} = C^{21} = 0$

[D] $C^{22} = C^{12} = 0$

Under reflection in the y-axis (i.e. interchange of z and $\bar{z}$), systems satisfying [A] are exchanged with those satisfying [C] and systems satisfying [B] are exchanged with solutions satisfying [D].

Note that we can use (8) – (20) to show that if one of these conditions holds at a point, then one of them must also hold in a neighborhood of that point. For example, if [A] holds at a point, then either [A] or [C] holds in a neighborhood. Hence, with out loss of generality, we can consider the conditions defining cases [A] to [D] as holding in a neighborhood.

3.1.1 [A] $C^{11} = C^{12} = 0$

With $C^{11} = C^{12} = 0$, equations (8) to (13) become

$$\begin{aligned} \frac{\partial C^{22}}{\partial z} &= 0 && (21)\\ \frac{\partial C^{22}}{\partial \bar{z}} &= \frac{2}{3}(C^{22})^2 - \frac{\partial C^{21}}{\partial z} && (22)\\ \frac{\partial C^{21}}{\partial \bar{z}} &= \frac{2}{3}C^{22}C^{21}\,. && (23) \end{aligned}$$

So C^{22} depends only on $\bar{z}$ and so we can set $C^{22} = X(\bar{z})$. Differentiating (22) with respect to z shows that C^{21} is at most linear in z, that is,

$$\frac{\partial^2 C^{21}}{\partial z^2} = 0 \qquad \Longrightarrow \qquad C^{21} = Y(\bar{z})z + Z(\bar{z}) \tag{24}$$

where $Y(\bar{z})$ and $Z(\bar{z})$ are functions to be determined. Now, (23) becomes

$$Y'(\bar{z})z + Z'(\bar{z}) = \frac{2}{3}(Y(\bar{z})z + Z(\bar{z}))X(\bar{z}) \tag{25}$$

and equating coefficients of z,

$$\begin{aligned} Y'(\bar{z}) &= \frac{2}{3}Y(\bar{z})X(\bar{z}) && (26)\\ Z'(\bar{z}) &= \frac{2}{3}Z(\bar{z})X(\bar{z})\,. && (27) \end{aligned}$$

First consider the case $Y(\bar{z}) = 0$. Then from (22), $X(\bar{z})$ satisfies

$$X'(\bar{z}) = \frac{2}{3}X(\bar{z})^2 \tag{28}$$

which has solutions

$$X(\bar{z}) = 0 \quad \text{or} \quad X(\bar{z}) = \frac{3}{c_1 - 2\bar{z}}\,, \tag{29}$$

where c_1 is a constant. Since we consider systems related by a translation to be equivalent, we can always set $c_1 = 0$. Using (27) we find two solutions

$$X(\bar{z}) = 0 \quad \Longrightarrow \quad Z(\bar{z}) = c_2 \tag{30}$$

and

$$X(\bar{z}) = -\frac{3}{2\bar{z}} \quad \Longrightarrow \quad Z(\bar{z}) = \frac{c_2}{\bar{z}}\,. \tag{31}$$

Note that rotations scale the each of C^{ij} by a nonzero factor, so we must distinguish the cases in which $c_2 = 0$ from those in which $c_2 \neq 0$. For the first solution (30) we consider $c_2 = 0$ and $c_2 = 2$.

1: $C^{11} = 0,\ C^{12} = 0,\ C^{22} = 0,\ C^{21} = 0$

$$V = \alpha z\bar{z} + \beta z + \gamma\bar{z} + \delta \qquad \text{(E3)}$$

2: $C^{11} = 0,\ C^{12} = 0,\ C^{22} = 0,\ C^{21} = 2$

$$V = \alpha\bar{z}\left(\bar{z}^2 + 3z\right) + \beta\left(\bar{z}^2 + z\right) + \gamma\bar{z} + \delta \qquad \text{(E10)}$$

For the second solution (31) we consider $c_2 = 0$ and $c_2 = \frac{3}{2}$.

3: $C^{11} = 0,\ C^{12} = 0,\ C^{22} = -\frac{3}{2\bar{z}},\ C^{21} = 0$

$$V = \frac{\alpha z}{\sqrt{\bar{z}}} + \frac{\beta}{\sqrt{\bar{z}}} + \gamma z + \delta \qquad \text{(E11)}$$

4: $C^{11} = 0,\ C^{12} = 0,\ C^{22} = -\frac{3}{2\bar{z}},\ C^{21} = \frac{3}{2\bar{z}}$

$$V = \frac{\alpha}{\sqrt{\bar{z}}} + \beta\left(z + \bar{z}\right) + \frac{\gamma(z + 3\bar{z})}{\sqrt{\bar{z}}} + \delta \qquad \text{(E9)}$$

Now, if $Y(\bar{z}) \neq 0$, then using (22) and (26) we find

$$\left(\frac{X(\bar{z})}{Y(\bar{z})}\right)' = \frac{X'(\bar{z})Y(\bar{z}) - X(\bar{z})Y'(\bar{z})}{Y(\bar{z})^2} = \frac{(\frac{2}{3}X(\bar{z})^2 - Y(\bar{z}))Y(\bar{z}) - X(\bar{z})(\frac{2}{3}X(\bar{z})Y(\bar{z}))}{Y(\bar{z})^2} = -1$$

$$\Longrightarrow \quad \frac{X(\bar{z})}{Y(\bar{z})} = c_1 - \bar{z} \quad \Longrightarrow \quad \frac{2Y(\bar{z})'}{3Y(\bar{z})^2} = c_1 - \bar{z} \quad \Longrightarrow \quad Y(\bar{z}) = \frac{3}{(\bar{z} - c_1)^2 - c_2}$$

¿From equation (26) we find $X(\bar{z})$ and equations (26) and (27) say that for $X(\bar{z}) \neq 0$, $Z(\bar{z})$ is a constant multiple of $Y(\bar{z})$,

$$X(\bar{z}) = -\frac{3\bar{z}}{(\bar{z} - c_1)^2 - c_2}, \qquad Z(\bar{z}) = \frac{c_3}{(\bar{z} - c_1)^2 - c_2}.$$

Again, we can remove c_1 and c_3 with a translation and, since c_2 is proportional to the relative invariant $(C^{22})^2 - 3C^{21}_z$, we must distinguish cases with $c_2 = 0$ from those with $c_2 \neq 0$. We consider the two solutions with $c_2 = 0$ and $c_2 = 1$.

5: $C^{11} = 0$, $C^{12} = 0$, $C^{22} = -\frac{3}{\bar{z}}$, $C^{21} = \frac{3z}{\bar{z}^2}$

$$V = \alpha z\bar{z} + \frac{\beta}{\bar{z}^2} + \frac{\gamma z}{\bar{z}^3} + \delta \qquad \text{(E8)}$$

6: $C^{11} = 0$, $C^{12} = 0$, $C^{22} = -\frac{3\bar{z}}{\bar{z}^2-1}$, $C^{21} = \frac{3z}{\bar{z}^2-1}$

$$V = \alpha z\bar{z} + \frac{\beta\bar{z}}{\sqrt{\bar{z}^2 - 1}} + \frac{\gamma z(2\bar{z}^2 - 1)}{\sqrt{\bar{z}^2 - 1}} + \delta \qquad \text{(E7)}$$

3.1.2 [B] $C^{11} = C^{21} = 0$

In this case, (8) and (19) become

$$\begin{aligned} 0 &= \frac{2}{3}C^{22}C^{12} - C^{12}_{\bar{z}} \\ 0 &= C^{22}(3C^{12}_{\bar{z}} + 2C^{12}C^{22}) \end{aligned}$$

and so, either $C^{22} = 0$, which then leads to system (E10) with z and $\bar{z}$ interchanged, or $C^{12} = 0$ which gives system (E11).

3.1.3 [C] $C^{22} = C^{21} = 0$ and [D] $C^{22} = C^{12} = 0$

Cases [C] and [D] give potentials obtained from those in cases [A] and [B] by interchanging z and $\bar{z}$.

3.2 $D \neq 0$

The remaining cases have $D \neq 0$. Using (18) and (19) we can solve for C^{21}_z and $C^{12}_{\bar{z}}$ to give

$$C^{12}_{\bar{z}} = \frac{\frac{2}{3}C^{12}\Big(4C^{21}(C^{11})^2 + C^{11}(C^{22})^2 + 3C^{22}C^{12}C^{21}\Big)}{9C^{12}C^{21} - C^{11}C^{22}} \tag{32}$$

$$C^{21}_{z} = \frac{\frac{2}{3}C^{21}\Big(4C^{12}(C^{22})^2 + C^{22}(C^{11})^2 + 3C^{11}C^{21}C^{12}\Big)}{9C^{12}C^{21} - C^{11}C^{22}}\,. \tag{33}$$

Substituting these into (20) we find the quartic identity becomes

$$\frac{N_1N_2N_3N_4}{D^2} = 0\,,$$

where

$$\begin{aligned}
N_1 &= C^{12}\,,\\
N_2 &= C^{21}\,,\\
N_3 &= C^{11}C^{22} - C^{12}C^{21}\,,\\
N_4 &= (C^{11})^2(C^{22})^2 - 27(C^{12})^2(C^{21})^2 + 4(C^{11})^3C^{21} + 4C^{12}(C^{22})^3 + 18C^{11}C^{22}C^{12}C^{21}\,.
\end{aligned}$$

It is easily verified that N_1 and N_2 scale under rotations, whereas D, N_3 and N_4 are invariant. From above, we can see that given $D \neq 0$, both partial derivatives of C^{12} are proportional to C^{12} and similarly for C^{21}. Hence, they are relative invariants. The same is true of N_3 and N_4.

$$\begin{aligned}
\frac{\partial N_3}{\partial z} &= \frac{2N_3((C^{11})^2C^{22} - 3C^{11}C^{21}C^{12} + 2C^{12}(C^{22})^2)}{3D}\\
\frac{\partial N_3}{\partial \bar{z}} &= \frac{2N_3((C^{22})^2C^{11} - 3C^{22}C^{12}C^{21} + 2C^{21}(C^{11})^2)}{3D}\\
\frac{\partial N_4}{\partial z} &= \frac{4}{3}N_4C^{11}\\
\frac{\partial N_4}{\partial \bar{z}} &= \frac{4}{3}N_4C^{22}
\end{aligned}$$

3.2.1 $N_1 = C^{12} = 0$

Since $D \neq 0$, we must have $C^{11} \neq 0$ and $C^{22} \neq 0$. The derivatives of the C^{11} become $\partial_z C^{11} = \frac{2}{3}(C^{11})^2$ and $\partial_{\bar{z}} C^{11} = 0$, hence, after a suitable translation,

$$C^{11} = -\frac{3}{2z}\,.$$

The other derivatives are now

$$\frac{\partial C^{22}}{\partial z} = 0 \tag{34}$$

$$\frac{\partial C^{22}}{\partial \bar{z}} = -\frac{2}{z}C^{21} + \frac{2}{3}(C^{22})^2 \tag{35}$$

$$\frac{\partial C^{21}}{\partial z} = \frac{1}{z}C^{21} \tag{36}$$

$$\frac{\partial C^{21}}{\partial \bar{z}} = \frac{2}{3}C^{22}C^{21}\,. \tag{37}$$

Equations (36) and (37) say that $C^{21} = X(\bar{z})z$, where $X(\bar{z})$ is a function to be determined. This leads to distinct cases when $X(\bar{z}) = 0$ or $X(\bar{z}) \neq 0$ since C^{21} is a relative invariant.

First consider $X(\bar{z}) = 0$, and so $C^{21} = 0$ and (35) has solution

$$C^{22} = -\frac{3}{2\bar{z}}\,. \tag{38}$$

As before, the constant of integration has been absorbed by a suitable translation. Note that in this case $N_4 \neq 0$. So we have

7: $C^{11} = -\frac{3}{2z}$, $C^{12} = 0$, $C^{22} = -\frac{3}{2\bar{z}}$, $C^{21} = 0$

$$V = \frac{\alpha}{\sqrt{z\bar{z}}} + \frac{\beta}{\sqrt{z}} + \frac{\gamma}{\sqrt{\bar{z}}} + \delta \qquad \text{(E20)}$$

Next consider $X(\bar{z}) \neq 0$ and so $C^{22} = 3X'(\bar{z})/(2X(\bar{z}))$. Substituting these into (35) gives

$$X''(\bar{z})X(\bar{z}) - 2X'(\bar{z})^2 + \frac{4}{3}X(\bar{z})^3 = 0 \quad \Rightarrow \quad \left(\frac{1}{X(\bar{z})}\right)'' = \frac{4}{3}$$

$$\Rightarrow \quad X(\bar{z}) = \frac{3}{2\bar{z}^2 + c_1\bar{z} - 2c_2}\,.$$

A translation can be used to set $c_1 = 0$. This solution gives

$$N_4 = \frac{81c_2}{4z^2(\bar{z}^2 - c_2)^2}\,,$$

which is a relative invariant and hence the case $c_2 \neq 0$ must be distinguished from $c_2 = 0$. We consider the two solutions with $c_2 = 0$ and $c_2 = 1$.

8: $C^{11} = -\frac{3}{2z}$, $C^{12} = 0$, $C^{22} = -\frac{3}{\bar{z}}$, $C^{21} = \frac{3z}{2\bar{z}^2}$

$$V = \frac{\alpha}{\sqrt{z\bar{z}}} + \frac{\beta}{\bar{z}\sqrt{z\bar{z}}} + \frac{\gamma}{\bar{z}^2} + \delta \qquad \text{(E17)}$$

9: $C^{11} = -\frac{3}{2z}$, $C^{12} = 0$, $C^{22} = -\frac{3\bar{z}}{\bar{z}^2-1}$, $C^{21} = \frac{3z}{2(\bar{z}^2-1)}$

$$V = \frac{\alpha\bar{z}}{\sqrt{\bar{z}^2 - 1}} + \frac{\beta}{\sqrt{z(\bar{z}+1)}} + \frac{\gamma}{\sqrt{z(\bar{z}-1)}} + \delta \qquad \text{(E19)}$$

3.2.2 $N_2 = C^{21} = 0$

Systems for which $N_2 = 0$ can be obtained from those with $N_1 = 0$ by reflection in the line $x = 0$, that is, by interchanging z and $\bar{z}$.

3.2.3 $N_3 = C^{11}C^{22} - C^{12}C^{21} = 0$

Since $D \neq 0$, all of the C^{ij} must be non vanishing and so we can make the substitution $C^{21} = C^{12}/(C^{11}C^{22})$. We can then show that

$$\frac{\partial^2}{\partial z^2}\left(\frac{C^{11}}{C^{22}}\right) = \frac{\partial^2}{\partial \bar{z}^2}\left(\frac{C^{22}}{C^{11}}\right) = 0 \quad \Longrightarrow \quad \frac{C^{11}}{C^{22}} = \frac{c_1 z + c_2}{c_3 \bar{z} + c_4}\,.$$

This observation allows the equations for the C^{ij} to be solved in a straight forward manner. Translations and rotations can be used to make specific choices for c_1, c_2, c_3, c_4, however, N_4 is proportional to c_1 and c_3 and it can be shown that $c_1 = 0 \Leftrightarrow c_3 = 0$. Hence we must distinguish cases in which $c_1 = c_3 = 0$ vanishes from those in which c_1 and c_2 are nonzero.

Firstly, with $c_1 = -1, c_2 = 0, c_3 = 1, c_4 = 0$, we find

10: $C^{11} = -\frac{3z}{z^2-\bar{z}^2}$, $C^{12} = \frac{3\bar{z}}{z^2-\bar{z}^2}$, $C^{22} = \frac{3\bar{z}}{z^2-\bar{z}^2}$, $C^{21} = -\frac{3z}{z^2-\bar{z}^2}$.

$$V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \delta \qquad \text{(E1)}$$

Secondly, with $c_1 = 0, c_2 = 1, c_3 = 0, c_4 = 1$, we find

11: $C^{11} = -\frac{3}{4x}$, $C^{12} = -\frac{3}{4x}$, $C^{22} = -\frac{3}{4x}$, $C^{21} = -\frac{3}{4x}$.

$$V = \alpha(x^2 + 4y^2) + \frac{\beta}{x^2} + \gamma y + \delta \qquad \text{(E2)}$$

3.2.4 $N_4 = 0$

The final possibility is that $N_4 = 0$ while N_1, N_2 and N_3 are nonzero. To simplify the equations, we note that N_4 can be written as

$$N_4 = \frac{1}{4}M_3 M_2^2 + M_1 M_4$$

where

$$\begin{aligned} M_1 &= C^{11} + C^{22} - C^{12} - C^{21} \\ M_2 &= C^{11} - C^{22} - 3C^{12} + 3C^{21} \\ M_3 &= 3C^{21}C^{12} - C^{11}C^{22} + C^{11}C^{21} + C^{22}C^{12} \end{aligned}$$

and M_4 is a cubic function of the C^{ij}. Since at least three of the C^{ij} are nonzero a rotation can be performed so that $M_1 = 0$ and furthermore, since

$$\begin{aligned}\frac{\partial M_1}{\partial z} &= M_5 M_1 - \frac{1}{2} M_2 M_3 (C^{11} - C^{22} + C^{12} - C^{21}) \\ \frac{\partial M_1}{\partial \bar{z}} &= M_6 M_1 - \frac{1}{2} M_2 M_3 (C^{11} - C^{22} + C^{12} - C^{21}),\end{aligned}$$

where M_5 and M_6 are quartic functions of the C^{ij}, the condition $M_1 = 0$ is maintained under translations when $N_4 = 0$. So the condition $N_4 = 0$ can be replaced by either $M_1 = M_2 = 0$, which leads to $N_3 = 0$ and so has already been considered, or $M_1 = M_3 = 0$. It can be shown that if $M_2 \neq 0$ and N_4 and M_1 vanish in a neighborhood, then so does M_3. So we can solve for C^{11} and C^{12} in terms of C^{22} and C^{21} and then, after a suitable translation the last remaining case is found.

12: $C^{11} = \frac{3(\bar{z}-2z)}{2(z(z-\bar{z}))}$, $C^{12} = \frac{3\bar{z}}{2(z(z-\bar{z}))}$, $C^{22} = \frac{3(z-2\bar{z})}{2(\bar{z}(\bar{z}-z))}$, $C^{21} = \frac{3z}{2(\bar{z}(\bar{z}-z))}$.

$$V = \frac{1}{\sqrt{x^2+y^2}} \left(\alpha + \frac{\beta}{x + \sqrt{x^2+y^2}} + \frac{\gamma}{x - \sqrt{x^2+y^2}} \right) + \delta \qquad \text{(E16)}$$

4 The quadratic algebra

The fact that the algebra of constants of the motion of a nondegenerate 2D superintegrable system closes quadratically was establish in [14] and has been used to determine the spectra of the quantum versions of these systems [21] as well as in their classification [19, 23]. Using results presented here we give an very direct and explicit description of the quadratic algebra.

Since we have an involutive system for the a_i, the dimension of the space of second order constants is three. At any point H has $(a_1^{(0)}, a_2^{(0)}, a_3^{(0)}) = (0, 4, 0)$. We can complete a basis by choosing two more second order constants, L_1 and L_2, which at a particular regular point have

$$(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}) = (1, 0, 0), \qquad (a_1^{(2)}, a_2^{(2)}, a_3^{(2)}) = (0, 0, 1).$$

Furthermore, we are free to choose the additive constant in V, $W^{(1)}$ and $W^{(2)}$ so that they vanish at the regular point. We denote the values of V_z, $V_{\bar{z}}$, and $V_{z\bar{z}}$ at the regular point by V_z^0, $V_{\bar{z}}^0$, and $V_{z\bar{z}}^0$.

If we define,

$$Q = \{L_1, L_2\},$$

then

$$
\begin{aligned}
Q^2 \;=\; & \frac{16}{9}\Big((C^{21})^2L_1^3 + (C^{11}C^{21} + 3C^{21}_z)L_1^2L_2 + (C^{22}C^{12} + 3C^{12}_{\bar z})L_1L_2^2 + (C^{12})^2L_2^3\Big) \\
& + \frac{4}{9}\Big(2(C^{22}C^{21}H + 3C^{21}V^0_{\bar z})L_1^2 + 2(C^{11}C^{12}H + 3C^{12}V^0_z)L_2^2 \\
& \qquad + ((5C^{12}C^{21} - C^{11}C^{22})H - 3C^{22}V^0_z - 3C^{11}V^0_{\bar z} + 9V^0_{z\bar z})L_1L_2\Big) \\
& + \frac{1}{9}\Big(((C^{22})^2 - 3C^{11}C^{21} - 3C^{21}_z)H^2 + 6(C^{22}V^0_{\bar z} - C^{21}V^0_z)H + 9(V^0_{\bar z})^2\Big)L_1 \\
& + \frac{1}{9}\Big(((C^{11})^2 - 3C^{22}C^{12} - 3C^{12}_{\bar z})H^2 + 6(C^{11}V^0_z - C^{12}V^0_{\bar z})H + 9(V^0_z)^2\Big)L_2 \\
& - \frac{1}{36}(C^{11}C^{22} + 7C^{12}C^{21})H^3 - \frac{1}{12}(C^{11}V^0_{\bar z} + C^{22}V^0_z + 3V^0_{z\bar z})H^2 - \frac{1}{2}V^0_{\bar z}V^0_zH\,.
\end{aligned}
$$

Since

$$
\{Q, L_1\} = -\frac{1}{2}\frac{\partial Q^2}{\partial L_2} \qquad \text{and} \qquad \{Q, L_2\} = \frac{1}{2}\frac{\partial Q^2}{\partial L_1}\,,
$$

equation (39) determines the quadratic algebra, and guarantees its existence.

Using discriminants of a cubic, it can be shown that the part of Q^2 that is cubic in L_1 and L_2 has at least a repeated factor. Cases in which the factor is a triple factor can be distinguished from those with a double factor. This analysis can be extended to classify the possible systems according to their quadratic algebra as they have been in [19] and [23].

5 Classification by invariants

We consider systems related by symmetries of the complex Euclidean plane to be equivalent. The local action of the group is given by (8) for the generators of translations. The action of the rotations is given by (7)

For each system there is a subgroup of motions that leaves the form of the potential unchanged, the isotropy subgroup of the system.

The relative invariants that distinguish each system from other systems as well as the generators of the isotropy subgroup of $E(2,\mathbb{C})$, are summarized in Tables 1 and 2.

In the case of systems with $D \neq 0$, it has been shown above that N_1, N_2, N_3 and N_4 are relative invariants and these are summarized in Table 1.

For the case $D = 0$, all systems found either have $C^{11} = C^{12} = C^{12}_{\bar z} = 0$ or can be obtained

	D	N_1	N_2	N_3	N_4	generators of the isotropy subgroup	
						continuous	discrete
E1	$\frac{9}{2x^2}+\frac{9}{2y^2}$	$\frac{-3i\bar{z}}{4xy}$	$\frac{3iz}{4xy}$	0	$-\frac{81}{4x^2y^2}$		$R_{x=0},\ R_{y=x}$
E2	$\frac{9}{2x^2}$	$-\frac{3}{4x}$	$-\frac{3}{4x}$	0	0	T_x	$R_{x=0},\ R_{y=0}$
E16	$\frac{9x^2}{2z\bar{z}y^2}$	$-\frac{3i\bar{z}}{4zy}$	$\frac{3iz}{4\bar{z}y}$	$\frac{9}{2z\bar{z}}$	0		$R_{x=0},\ R_{y=0}$
E17	$-\frac{9}{2z\bar{z}}$	0	$\frac{3z}{2\bar{z}^2}$	$\frac{9}{2z\bar{z}}$	0	M_θ	
E19	$-\frac{9\bar{z}}{2(\bar{z}^2-1)}$	0	$\frac{3z}{2(\bar{z}^2-1)}$	$\frac{9\bar{z}}{2z(\bar{z}^2-1)}$	$\frac{81}{z^2(\bar{z}^2-1)^2}$		M_π
E20	$-\frac{9}{4z\bar{z}}$	0	0	$\frac{9}{4z\bar{z}}$	$\frac{81}{16z^2\bar{z}^2}$	M_θ	$R_{x=0}$

Table 1: Relative invariants and generators of the isotropy subgroup for systems with $D \neq 0$.

	D	$(C^{22})^2-3C^{21}_z$	C^{21}_z	C^{21}	generators of the isotropy subgroup	
					continuous	discrete
E3	0	0	0	0	$T_x,\ T_y,\ M_\theta$	
E7	0	$\frac{9}{(\bar{z}^2-1)^2}$	$\frac{3}{\bar{z}^2-1}$			M_π
E8	0	0	$\frac{3}{\bar{z}^2}$		M_θ	
E9	0	$\frac{9}{4\bar{z}^2}$	0	$\frac{3}{2\bar{z}}$	T_x+iT_y	M_π
E10	0	0	0	2	T_x, T_y	
E11	0	$\frac{9}{4\bar{z}^2}$	0	0	$T_x+iT_y,\ M_\theta$	

Table 2: Relative invariants and generators of the isotropy subgroup for systems with $D=0$, assuming $C^{11}=C^{12}=C^{12}_{\bar{z}}=0$.

from these systems by $y \leftrightarrow -y$ and so we assume $C^{11} = C^{12} = C^{12}_{\bar z} = 0$ and find

$$\begin{aligned}
\partial_z C^{21}_z &= 0 \\
\partial_{\bar z} C^{21}_z &= \frac{2}{3} C^{22} C^{21}_z \\
\partial_z \left((C^{22})^2 - 3C^{21}_z \right) &= 0 \\
\partial_{\bar z} \left((C^{22})^2 - 3C^{21}_z \right) &= \frac{4}{3} C^{22} \left((C^{22})^2 - 3C^{21}_z \right) .
\end{aligned}$$

Thus, since C^{21}_z and $(C^{22})^2 - 3C^{21}_z$ scale under rotations, they are relative invariants in case [A] above. Further, if $C^{21}_z = 0$, then $\partial_z C^{21} = 0$ and $\partial_{\bar z} C^{21} = \frac{2}{3} C^{22} C^{21}$ and hence it is a relative invariant in case [A] with $C^{21}_z = 0$.

Note that the two tables show that the isotropy subgroups, together with the condition $D \equiv 0$ or $D \neq 0$, are sufficient to distinguish all superintegrable systems.

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