Name: Master

Math 4242. Section 10 Final Exam May 13, 2003
(There are a total of 200 points on this exam)

Problem 1 *Consider the system of equations*

$$\begin{aligned} 5x_1 + 4x_2 + 7x_3 + 10x_4 &= 100 \\ 20x_1 + 25x_2 + 10x_3 + 5x_4 &= 200 \\ 2x_1 + 2x_2 + 10x_3 + 6x_4 &= 50. \end{aligned}$$

a. (15 points) *Find the general solution of this system.*

Reduce augmented matrix to REF

$$\left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 25 \\ & 1 & -18 & -11 & -60 \\ & & 36 & 16 & 85 \end{array}\right)$$

free variable

Use back substitution to get general soln:

$$x_3 = \frac{1}{36}(85 - 16x_4)$$

$$x_2 = -\frac{35}{2} + 3x_4$$

$$x_1 = \frac{85 \cdot 13}{36} - \frac{34}{9}x_4$$

← not necessary to get x_1 by back substitution, if pressed for time

b. (5 points) *A veterinarian recommends that a certain pet's diet should contain 100 units of protein, 200 units of carbohydrates, and 50 units of fat daily. A store's pet food department contains four varieties of foods with the following composition of protein, carbohydrate, and fat (in units) per ounce in foods A, B, C, and D:*

Food	Protein	Carbohydrates	Fat
A	5	20	2
B	4	25	2
C	7	10	10
D	10	5	6

Our problem is to find, if possible, the amounts of foods A, B, C, and D that can be included in the pet's diet to conform to the veterinarian's recommendation. First, show the relationship between this problem and the above linear system.

x_1 = ounces of food of type A in pet's diet

x_2 = B

x_3 = C

x_4 = D

protein $5x_1 + 4x_2 + 7x_3 + 10x_4 = 100$

carbs. $20x_1 + 25x_2 + 10x_3 + 5x_4 = 200$

fat $2x_1 + 2x_2 + 10x_3 + 6x_4 = 50$

c. (5 points) *What is your answer to the question of finding the amounts of foods A, B, C, and D that can be included in the pet's diet to conform to the veterinarian's recommendation?*

For acceptable soln. must have $x_1, x_2, x_3, x_4 \geq 0$

$$\left.\begin{array}{l} x_3 \geq 0 \Rightarrow x_4 \leq \frac{85}{16} \\ x_2 \geq 0 \Rightarrow x_4 \geq \frac{35}{6} \end{array}\right\} \Rightarrow 2\ \frac{35}{6} \leq x_4 \leq \frac{84}{16} \Longleftrightarrow \frac{280}{48} \leq x_4 \leq \frac{255}{48}$$

Impossible!

No soln.

Problem 2

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 0 & 5 \end{pmatrix}.$$

a. (15 points) *Find a PLU-decomposition of A.*

$$A \underset{E_{14}(-1)E_{12}(1)E_{34}(-4)E_{23}(-1)}{\sim} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -13 & 3 \end{pmatrix} \overset{P_{34}}{\sim} \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & -13 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

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$$A = \underbrace{\overset{4\times 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}}_{P} \underbrace{\overset{4\times 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}}}_{L} \underbrace{\overset{4\times 5}{\begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & -13 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}}}_{U}$$

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b. (5 points) *What is the rank of A?*

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c. (10 points) *Find a basis for the range (column space) of A.*

3 Column 3 is free, so column vector 1,2,4,5 form a basis.

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 5 \end{pmatrix}$$

d. (5 points) *What is the dimension of the null space of A?*

$A_{m \times n}$ $\quad m=4, \; n=5$

$$\dim \mathcal{N}(A) + \underbrace{\dim R(A)}_{= \dim R(A^T) = 4} = n = 5$$

$$\Rightarrow \dim \mathcal{N}(A) = 1$$

e. (5 points) *What is the dimension of the range of A^{T}?*

$$\dim R(A) = \dim R(A^T) = 4$$

f. (5 points) *What is the relationship between the null space of A and the range of A^{T}?*

$$\mathcal{R}(A^T)^{\perp} = \mathcal{N}(A)$$

$$\mathcal{N}(A)^{\perp} = \mathcal{R}(A^T)$$

Problem 3 (15 points) *Use the Gram-Schmidt process to find an ON basis for the span of*

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix},$$

in R^4.

$\|a_1\|^2 = 4$, $\quad \boxed{v_1 = \frac{1}{2} a_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}} \quad v_1^T a_2 = 1$

$b_2 = a_2 - (v_1^T a_2) v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$, $\quad \|b_2\|^2 = 3$

$\boxed{v_2 = \frac{1}{\sqrt{3}} b_2 = \frac{\sqrt{3}}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}}$

$(v_1^T a_3) = 1$

$(v_2^T a_3) = -\frac{1}{3}$

$b_3 = a_3 - (v_1^T a_3) v_1 - (v_2^T a_3) v_2$

$= \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{2} \end{pmatrix} = \frac{4}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$

$\|b_3\|^2 = \frac{8}{3}$

$\boxed{v_3 = \frac{\sqrt{6}}{6} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}}$

Problem 4 *Consider planes of the form*

$$z = ax + by + c.$$

We want to find the least squares fit for a plane through the five points

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

a. (5 points) *Write the system of equations to be fit in the form $Av = d$ where*

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

What are A and d?

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}}_{= A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 2 \end{pmatrix}}_{= d}$$

b. (15 points) *Determine the least squares fit for $Av = d$*

$A^T A v = A^T d$, $A^T A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 5 \end{pmatrix}$, $A^T d = \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 0 & 3 & 6 \\ 0 & 3 & 1 & 1 \\ 3 & 1 & 5 & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -5 & 1 \end{array}\right)$$

back substitution: $c = -\frac{1}{5}$

$b = \frac{2}{5}$

$a = \frac{11}{5}$

c. (5 points) *What is the equation of the plane that best fits the data?*

$Z = \frac{11}{5}X + \frac{2}{5}Y - \frac{1}{5}$ or $11X + 2Y - 5Z = 1$

Problem 5 *Answer each of the following as true or false. (The matrices are all real $n \times n$). If true, give a reason; if false give a counterexample.*

a. (5 points) $\det(AA^{\mathrm{T}}) = \det(A^2)$. true

$$\det(AA^T) = \det A \det A^T = \det A \det A = \det(A^2)$$

b. (5 points) *If* $\det(A) > 0$ *then A has no negative eigenvalues.* false

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \det A = 1$$

eigenvalues $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$

c. (5 points) *If* $\det(A) = 0$ *then* $\det(\operatorname{adj} A) = 0$. true

$$A \operatorname{adj} A = (\det A) I$$

$\Rightarrow A \operatorname{adj} A = Z$ = zero matrix

If $\det(\operatorname{adj} A) \neq 0$ the $\operatorname{adj} A$ is nonsingular,

so $A = A \operatorname{adj} A \cdot (\operatorname{adj} A)^{-1} = Z (\operatorname{adj} A)^{-1} = Z$

$\Rightarrow A = Z \Rightarrow \operatorname{adj} A = Z$, a contradiction!

$\therefore \det(\operatorname{adj} A) = 0$

d. (5 points) *If $A = SBS^{-1}$ and S is nonsingular, then* $\det(A) = \det(B)$.

true

$$\det A = \det(SBS^{-1}) = \det S \det B \det S^{-1}$$
$$= (\det S)(\det S)^{-1} \det B = \det B$$

Problem 6 *Two masses are adjoined by springs and the ends A and B are fixed. (See the figure on the board.) The masses are free to move horizontally. We assume that the three springs are uniform and that initially the system is in the equilibrium position. A force is exerted on the system to set the masses in motion. The horizontal displacement of the masses at time t will be denoted by $x_1(t)$ and $x_2(t)$, respectively. We assume that there are no frictional retarding forces. Then the only forces acting on mass m_1 at time t will be from springs 1 and 2. The force from spring 1 will be $-kx_1$ and the force from spring 2 will be $k(x_2 - x_1)$. By Newton's second law,*

$$m_1 x_1''(t) = -kx_1 + k(x_2 - x_1).$$

Similarly the only forces acting on the second mass will be from springs 2 and 3. From Newton's second law we have

$$m_2 x_2''(t) = -k(x_2 - x_1) - kx_2.$$

In the case where $m_1 = m_2 = 1$, $k = 1$ and assuming that the initial velocity of both masses is +2 units per second we have the system

$$\frac{d^2x(t)}{dt^2} = Ax(t), \qquad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

with initial conditions

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad x'(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

a. (10 points) *Compute the eigenvalues and eigen vectors of the matrix A.*

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 = (2+\lambda)^2 - 1 = (\lambda+3)(\lambda+1)$$

$$\Rightarrow \boxed{\lambda_1 = -3, \ \lambda_2 = -1}$$

$$\lambda_1 = -3: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x + y = 0, \ \boxed{v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -x + y = 0, \ \boxed{v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} u(t) + w(t) \\ -u(t) + w(t) \end{pmatrix}$$

b. (5 points) *Write down the general solution for the system* $\frac{d^2x(t)}{dt^2} = Ax(t)$.

$$x(t) = u(t)v_1 + w(t)v_2 \Leftrightarrow \ddot{u} v_1 + \ddot{w} v_2 = uAv_1 + wAv_2$$

$$= -3uv_1 - wv_2$$

$$\Leftrightarrow \ddot{u} + 3u = 0, \ \ddot{w} + w = 0$$

$$u(t) = \alpha_1 \cos\sqrt{3}t + \alpha_2 \sin\sqrt{3}t, \ w(t) = \beta_1 \cos t + \beta_2 \sin t$$

$$x(t) = \begin{pmatrix} \alpha_1 \cos\sqrt{3}t + \alpha_2 \sin\sqrt{3}t + \beta_1 \cos t + \beta_2 \sin t \\ -\alpha_1 \cos\sqrt{3}t - \alpha_2 \sin\sqrt{3}t + \beta_1 \cos t + \beta_2 \sin t \end{pmatrix}$$

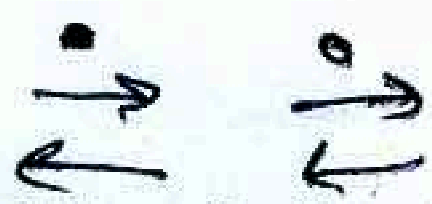
c. (10 points) *Find the special solution of the system that satisfies the initial conditions and give a physical interpretation of the answer.*

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 \\ -\alpha_1 + \beta_1 \end{pmatrix} \Rightarrow \alpha_1 = \beta_1 = 0$$

$$\dot{x}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}\,\alpha_2 + \beta_2 \\ -\sqrt{3}\,\alpha_2 + \beta_2 \end{pmatrix} \Rightarrow \beta_2 = 2, \quad \alpha_2 = 0$$

Solution: $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 2 \sin t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The masses are vibrating exactly in phase and with amplitude 2 & frequency 1.



Problem 7 *Consider the quadratic form*

$$g(x) = -2x_1^2 + 4x_2^2 + 6x_1x_2 - 4x_3^2, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

a. (5 points) *Find the real symmetric matrix A such that $g(x) = x^{\mathrm{T}} A x$.*

$$g(x) = (x_1, x_2, x_3) \underbrace{\begin{pmatrix} -2 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix}}_{= A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

b. (10 points) *Diagonalize the quadratic form by means of some orthogonal matrix. (You need compute only the simplified (diagonal) form, not the orthogonal matrix.)*

$$\begin{vmatrix} -2-\lambda & 3 & 0 \\ 3 & 4-\lambda & 0 \\ 0 & 0 & -4-\lambda \end{vmatrix} = 0 = (4+\lambda)\Big((2+\lambda)(4-\lambda)+9\Big)$$
$$= -(4+\lambda)(\lambda^2-2\lambda-17)$$

$\lambda_1 = -4,\ \lambda_2 = 1+3\sqrt{2},\ \lambda_3 = 1-3\sqrt{2}$

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & 1+3\sqrt{2} & 0 \\ 0 & 0 & 1-3\sqrt{2} \end{pmatrix} \qquad -4{x'_1}^2 + (1+3\sqrt{2}){x'_2}^2 + (1-3\sqrt{2}){x'_3}^2$$

c. (5 points) *Is this quadratic form positive definite, positive semidefinite or indefinite?*

indefinite, because $\lambda_2 > 0$, $\lambda_3 < 0$
$\lambda_1 < 0$

Problem 8 *Let $T : R^3 \longrightarrow R^3$ be the linear transformation $T(x) = Ax$, where*

$$\overset{①}{T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}}, \quad \overset{②}{T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\5\\-4\end{pmatrix}}, \quad \overset{③}{T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}6\\8\\-10\end{pmatrix}}.$$

a. (10 points) *Compute the 3×3 matrix A.*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$① \Rightarrow \boxed{\begin{matrix} a_{11} = 1 \\ a_{21} = 2 \\ a_{31} = -1 \end{matrix}}$$

$$② \Rightarrow \begin{matrix} a_{11} + a_{12} = 3 \\ a_{21} + a_{22} = 5 \\ a_{31} + a_{32} = -4 \end{matrix} \Rightarrow \boxed{\begin{matrix} a_{12} = 2 \\ a_{22} = 3 \\ a_{32} = -3 \end{matrix}}$$

$$③ \Rightarrow \begin{matrix} a_{12} + a_{13} = 6 \\ a_{22} + a_{23} = 8 \\ a_{32} + a_{33} = -10 \end{matrix} \Rightarrow \boxed{\begin{matrix} a_{13} = 4 \\ a_{23} = 5 \\ a_{33} = -7 \end{matrix}}$$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ -1 & -3 & -7 \end{pmatrix}$$

b. (5 points) *Find an equation relating $a, b,$ and c so that $\begin{pmatrix}a\\b\\c\end{pmatrix}$ will lie in the range of T.*

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 2 & 3 & 5 & b \\ -1 & -3 & -7 & c \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 2 & 4 & a \\ 0 & -1 & -3 & b-2a \\ 0 & -1 & -3 & a+c \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 2 & 4 & a \\ 0 & \boxed{-1} & -3 & b-2a \\ 0 & 0 & 0 & 3a-b+c \end{array}\right)$$

$$\boxed{\therefore \begin{pmatrix}a\\b\\c\end{pmatrix} \in R(T) \Longleftrightarrow 3a-b+c=0}$$

Alternate soln:

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}$$

$$= B^H B$$

Problem 9 *(10 points) Given*

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

find a square matrix B such that $A = B^H B$.

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 1-\lambda & i \\ 0 & -i & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(4-\lambda)(\lambda-2)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4$$

normalization

$$\lambda_1 = 0: \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x = 0 \\ z = iy \end{matrix}, \quad \boxed{w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}}$$

$$\lambda_2 = 2: \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & i \\ 0 & -i & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x = 0 \\ z = -iy \end{matrix} \quad w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

$$\lambda_3 = 4: \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & i \\ 0 & -i & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} y = z = 0 \\ x \text{ arb.} \end{matrix}, \quad w_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Set $\underset{\text{unitary}}{U} = (w_1, w_2, w_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ -i & i & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$AU = U\Lambda$, $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $\Lambda^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Set $B = U\Lambda^{\frac{1}{2}}U^H = \frac{1}{2} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ -i & i & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & i & 1 \\ 0 & -i & 1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2, & 0, & 0 \\ 0, & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then $B = B^H$ and

$$A = B^H B = B^2$$