$\underset{n\times n}{A}$, $\underset{A}{P}(\lambda) = (-1)^n (\lambda - \lambda_1)^{l_1} \cdots (\lambda - \lambda_k)^{l_k}$

characteristic polynomial $\lambda_1, \ldots, \lambda_k$ distinct

If $\dim V_{\lambda_i} = l_i$ then the eigenspace V_{λ_i} is complete

A is complete (diagonalizable) $\iff$ each eigenspace V_{λ_i} of A is complete

Ex: $\underset{2\times 2}{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\underset{A}{P}(\lambda) = (\lambda - 1)^2$
$\lambda = 1, 1$

$V_1 = \left\{ \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ V_1 is not complete

Can't diagonalize A.

$\underset{n\times n}{A} = \begin{pmatrix} a & 1 & & 0 \\ & a & \ddots & \\ & & \ddots & 1 \\ 0 & & & a \end{pmatrix}$ Jordan block , $P_A(\lambda) = (-1)^n (\lambda - a)^n$
$\lambda = \underbrace{a, \ldots, a}_{n \text{ times}}$

$V_a = \left\{ \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \right\}$ V_a is not complet.

Some properties of similar matrices

$\underset{n\times n}{A}, \underset{n\times n}{B}$ Suppose $B = S^{-1}AS$ for $\underset{n\times n}{S}$ nonsingular

$$\underset{\text{lin. op.}}{L} : V \to V \begin{array}{l} \nearrow \text{basis } \{\underset{\sim}{w}_1, \dots, \underset{\sim}{w}_n\} \quad A \\ \searrow \text{basis } \{\underset{\sim}{z}_1, \dots, \underset{\sim}{z}_n\} \quad B \end{array} \qquad \underset{\sim}{z}_i = \sum_j S_{ji} \underset{\sim}{w}_j$$

$$\begin{aligned} P_B(\lambda) &= \det(B - \lambda I_n) = \det(S^{-1}AS - \lambda S^{-1}S) \\ &= \det(S^{-1}[A - \lambda I_n]S) \\ &= \det S^{-1} \det(A - \lambda I_n) \det S = \det(A - \lambda I_n) \\ &= P_A(\lambda) \end{aligned}$$

$\therefore P_B(\lambda) = P_A(\lambda) \Rightarrow$ A & B have the same eigenvalues with the same multiplicities

Also: $\det B = \det A$, $\operatorname{tr} B = \operatorname{tr} A$

In general:

$\underset{n\times n}{A}$ symmetric, real $\Rightarrow$ complete

$$A = Q \Lambda Q^T, \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ real}$$

$$\begin{aligned} q(\underset{\sim}{x}) = \underset{\sim}{x}^T A \underset{\sim}{x} &= \underset{\sim}{x}^T Q \Lambda Q^T \underset{\sim}{x} \\ &= \underset{\sim}{y}^T \Lambda \underset{\sim}{y} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \end{aligned}$$

Set $\boxed{\begin{array}{l} \underset{\sim}{y} = Q^T \underset{\sim}{x} \\ \underset{\sim}{y}^T = \underset{\sim}{x}^T Q \end{array}}$

Theorem: A is pos. def. $\Leftrightarrow \lambda_1, \lambda_2, \dots, \lambda_n > 0$

A is semi-pos. def $\Leftrightarrow \lambda_1, \lambda_2, \dots, \lambda_n \geq 0$.

Ex: $A = \begin{pmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$

$$p_A(\lambda) = (3-\lambda)(2-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1)$$

$$\lambda = 1: \quad \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

ON basis: $\underset{\sim}{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

$$\lambda = 4: \quad \begin{pmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

ON basis: $\underset{\sim}{u}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

$$Q = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

orthogonal

$$A = Q\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}Q^T$$

Set $\underset{\sim}{y} = Q^T\underset{\sim}{x}$, $\underset{\sim}{y}^T = \underset{\sim}{x}^T Q$

$$\Rightarrow q(\underset{\sim}{x}) = \underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{y}^T\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\underset{\sim}{y}$$

$$= 1y_1^2 + 4y_2^2, \quad \text{positive def.}$$

Proof of main theorem:

$\underset{n\times n,\ \text{real}}{A}$, $A^T = A$ eigenspaces V_{λ_i}

$\lambda_1, \dots, \lambda_k$ distint eigenvalues

$\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$, Euclidean inner product

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \underset{\sim}{x}^T \underset{\sim}{y}$$

1) Claim $\langle A\underset{\sim}{x}, \underset{\sim}{y} \rangle = \langle x, A\underset{\sim}{y} \rangle$, all $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^3$

Proof: $\langle A\underset{\sim}{x}, \underset{\sim}{y} \rangle = (A\underset{\sim}{x})^T \underset{\sim}{y} = \underset{\sim}{x}^T A^T \underset{\sim}{y}$

$= x^T (A\underset{\sim}{y}) = \langle x, A\underset{\sim}{y} \rangle$

2) λ_i is real, i.e. $\overline{\lambda_i} = \lambda_i$

Proof: Let $\underset{\sim}{v} \neq \underset{\sim}{0}$ such that $A\overline{\underset{\sim}{v}} = \overline{\lambda_i}\overline{\underset{\sim}{v}}$

Note: a) $\underset{\sim}{v}$ is an eigenvector
b) $\underset{\sim}{v}$ may be complex
c) $\overline{A} = A$

Consider $\langle A\overline{v}, v \rangle = \langle \overline{v}, A\underset{\sim}{v} \rangle = \lambda_i \langle \overline{v}, v \rangle$
$= \lambda_i \sum |v_j|^2$

$\underset{\sim}{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\quad \overline{\lambda_i} \langle \overline{v}, v \rangle = \overline{\lambda_i} \underbrace{\sum_{j=1} |v_j|^2}_{\neq 0}$

$\Rightarrow \boxed{\lambda_i = \overline{\lambda_i}}$

3) $V_{\lambda_1} \perp V_{\lambda_2}$ if $\lambda_1 \neq \lambda_2$

Proof: Take $\underset{\sim}{v_1}, \underset{\sim}{v_2}$ ~~nonzero~~ such that $A\underset{\sim}{v_i} = \lambda_i \underset{\sim}{v_i}$

$i = 1, 2$

Then $\langle A\underset{\sim}{v}_1, \underset{\sim}{v}_2\rangle = \langle \underset{\sim}{v}_1, A\underset{\sim}{v}_2\rangle = \lambda_2 \langle \underset{\sim}{v}_1, \underset{\sim}{v}_2\rangle$

$\quad \shortparallel$

$\lambda_1 \langle \underset{\sim}{v}_1, \underset{\sim}{v}_2\rangle$

$\Rightarrow (\lambda_2 - \lambda_1)\langle \underset{\sim}{v}_1, \underset{\sim}{v}_2\rangle = 0$

$\Rightarrow \langle \underset{\sim}{v}_1, \underset{\sim}{v}_2\rangle = 0$

4) A is complete

Proof: Form $W = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k} \in \mathbb{R}^n$

An ON basis for W can be made up from an ON basis of each eigenspace V_{λ_i} as λ_i runs over all eigenvalues of A.

$W \oplus W^{\perp} = \mathbb{R}^n$

Claim $W^{\perp} = \{\underset{\sim}{0}\}$ so $\mathbb{R}^n = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$.

Main idea: Assume $W^{\perp} \neq \{\underset{\sim}{0}\}$

Choose basis for $W^{\perp}$.

Let $\underset{\sim}{z} \in W^{\perp}$, $\underset{\sim}{w} \in W$, so $\langle \underset{\sim}{z}, \underset{\sim}{w}\rangle = 0$.

$\langle A\underset{\sim}{z}, \underset{\sim}{w}\rangle = \langle \underset{\sim}{z}, \underbrace{A\underset{\sim}{w}}_{\in W}\rangle = 0$

so $A\underset{\sim}{z} \perp W$

$\Rightarrow A\underset{\sim}{z} \in W^{\perp}$. $A: W^{\perp} \to W^{\perp}$

$\Rightarrow$ A has an eigenvector in $W^{\perp}$. Impossible!

$\Rightarrow W^{\perp} = \{\underset{\sim}{0}\}$