Name: ______________________________

Math 4242/4457 Sec. 10 Midterm Exam I October 26, 2007

There are a total of 100 points on this 55 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and ONE 8.5×11 inch sheet of notes are allowed, but no books, other notes, cell phones or other elecronic devices are allowed. Do all of your calculations on this test paper.

Problem	Score
1.	________
2.	________
3.	________
4.	________
5.	________
6.	________
Total:	________

Problem 1 *(10 points) Is each of these statements true or false? Give a brief justification of your answer, or a counterexample .*

(a) True or false: Every homogeneous linear system has a solution.

Solution: True. The equation $A\mathbf{x} = \mathbf{0}$ *always has the solution* $\mathbf{x} = \mathbf{0}$.

(b) True or false: If det$(A^2) = 1$ *for an* $n \times n$ *matrix* A *then* A *is necessarily nonsingular.*

Solution: True.

$$(\det\ A)^2 = \ \det\ (A^2) = 1,$$

so $\det\ A \neq 0$*, and* A *is nonsingular.*

Problem 2

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

a. (5 points) *Determine the rank of* A.

Solution: The row echelon form for A *is*

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix}.$$

There are 3 pivots, so the rank is $r = 3$.

b. (5 points) *Determine the dimensions of ker(A) and of coker(A).*

Solution: $m = 3,\ n = 5$. *dim ker* $(A) = n - r = 5 - 3 = 2$. *dim coker* $(A) = m - r = 3 - 3 = 0$.

c. (5 points) *Find a basis for ker(A).*

Solution: U has 3 free columns. Solving the homogeneous equation $U\mathbf{z} = \mathbf{0}$ *we find the basis*

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

d. (5 points) *Find the general solution of the equation*

$$Ax = b, \qquad b = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}.$$

Solution: The augmented matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 5 & \vdots & -3 \\ 0 & 0 & 1 & 2 & -1 & \vdots & 1 \\ 0 & 1 & 0 & 0 & -1 & \vdots & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & \vdots & 3 \\ 0 & 0 & 1 & 2 & 5 & \vdots & -3 \\ 0 & 0 & 0 & 0 & -6 & \vdots & 4 \end{pmatrix}.$$

A particular solution is $\mathbf{x}_P$ *where* $\mathbf{x}_p^T = (0, 7/3, 1/3, 0, -2/3)$ *and the general solution is*

$$\mathbf{x} = \mathbf{x}_P + \mathbf{z}_G = \begin{pmatrix} 0 \\ 7/3 \\ 1/3 \\ 0 \\ -2/3 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 3 *A 2×2 matrix A is a semi-magic square if its row sums and column sums all add up to the same number, e.g.*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

As you know from a homework problem, the set of 2×2 semi-magic squares forms a vector space.

(a) *(15 points) What is the dimension of this vector space? Find a basis.*

Solution: Let

$$A = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$$

be a semi-magic square. Then

$$\begin{aligned} x + y - z - u &= 0 \quad (1) \\ x + z - z - u &= 0 \quad (2) \\ y + u - z - u &= 0 \quad (3) \end{aligned}$$

so $x = u$, $y = z$, i.e. the system of equations has 2 pivots and 2 free variables z, u. Thus the general solution is

$$A = \begin{pmatrix} u & z \\ z & u \end{pmatrix} = z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The dimension is 2 and a basis is

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(b) *(5 points) A symmetric 3×3 semi-magic square is a matrix A such that $A = A^T$ and the row and column sums add up to the same number, e.g.,*

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

What is the dimension of the space of 3×3 symmetric semi-magic squares?

Solution: A general symmetric semi-magic square is

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

where

$$\begin{aligned} a+b+c &= b+d+e && (4) \\ c+e+f &= b+d+e. && (5) \end{aligned}$$

Thus there are just 2 independent equations, so 2 pivots and 4 free columns. It follows that the dimension is 4.

Problem 4 *(20 points) For what range of numbers a is the matrix*

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 2 & a & 8 \\ 4 & 8 & 7 \end{pmatrix}$$

positive definite?

Solution: Putting A in row echelon form we find

$$U = \begin{pmatrix} 2 & 2 & 4 \\ 0 & a-2 & 4 \\ 0 & 0 & -1-\frac{16}{a-2} \end{pmatrix}.$$

The condition that A be positive definite is that all pivots are positive, which implies both $a > 2$ and $a < -14$. This is impossible, so A is never positive definite.

Problem 5

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix}.$$

(15 points) *Compute the LU factorization of A.*

Solution:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 6 (15 points). *Set up the equations for a least squares solution to the inconsistent system* $A\mathbf{x} = \mathbf{b}$ *where*

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Do not solve the system. Will the solution necessarily be unique? Why?

Solution:

$$A^T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The system is $A^T A\mathbf{x} = A^T\mathbf{b}$ *or*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution is unique because the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ *is nonsingular.*