Name: ______________________________

Math 4242/4457 Sec. 10 Practice Final Exam

There are a total of 175 points on this 120 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and ONE 8.5×11 inch sheet of notes are allowed, but no books, other notes, cell phones or other electronic devices are allowed. Do all of your calculations on this test paper.

Problem	Score
1.	________
2.	________
3.	________
4.	________
5.	________
6.	________
7.	________
8.	________
9.	________
Total:	________

Problem 1 *Let $T : P_2 \longrightarrow R^2$ be the linear transformation from the space of polynomials $p(t) = at^2 + bt + c$ to R^2 defined by*

$$T(p(t)) = \begin{pmatrix} p(0) \\ \frac{dp(1)}{dt} \end{pmatrix}.$$

a. (10 points) *Compute the dimension of the kernel (null space) of T.*

b. (10 points) *Compute the dimension of the range of T.*

Problem 2 *Note that the individual parts of this two-part problem are closely related. You can use the results of part a. to solve part b. , if you wish.*

a. (10 points) *The vectors*

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

form a basis for a two-dimensional subspace V of R^3. Use the Gram-Schmidt Process on these vectors to obtain an orthonormal basis for V.

b. (10 points) *Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$ where*

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Is the solution unique?

Problem 3 *Suppose the singular value decomposition of the* 3×4 *matrix* A *is* $A = P\Sigma Q^{\mathrm{T}}$ *where*

$$P = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

a. (5 points) *Find the eigenvalues of* $A^{\mathrm{T}}A$.

b. (5 points) *Find a basis for the kernel of* A.

c. (5 points) *Find a basis for the range of* A.

d. (5 points) *Find a singular value decomposition of* $-A^{\mathrm{T}}$.

Problem 4

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 1 & 2 & 5 & 0 & 5 \\ 0 & 1 & 1 & 3 & 4 \end{pmatrix}.$$

a. (10 points) *Find an LU-decomposition of A.*

b. (4 points) *Find a basis for the range (column space) of A.*

c. (4 points) *Find a basis for the kernel of* A

d. (2 points) *What is the dimension of the cokernel of* A*?*

Problem 5 *Let*

$$A = \begin{pmatrix} 0 & 0 & 4 \\ -2 & -4 & -2 \\ -2 & 0 & 6 \end{pmatrix}.$$

a. (10 points) *Compute the eigenvalues of A.*

b. (10 points) *Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.*

Problem 6 *Consider the quadratic form*

$$p(\mathbf{x}) = -2x_1^2 + 4x_2^2 + 6x_1x_2 - 4x_3^2, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

a. (5 points) *Find the real symmetric matrix K such that $p(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} K \mathbf{x}$.*

b. (10 points) *Diagonalize the quadratic form by means of some orthogonal matrix. (You need compute only the simplified form, not the orthogonal matrix.)*

c. (5 points) *Is this quadratic form positive definite, positive semidefinite or indefinite?*

Problem 7 a. (10 points) *Suppose we are looking for an $m \times n$ matrix A and column vectors $\mathbf{b}$ and $\mathbf{c}$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions and $A^{\mathrm{T}}\mathbf{y} = \mathbf{c}$ has exactly one solution. Why is it impossible to find $A, \mathbf{b}, \mathbf{c}$?*

b. (5 points) *Suppose you are given a vector $\mathbf{b}$, a vector $\mathbf{p}$ and n linearly independent vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n$, all in the same inner product space V. If I claim that $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the subspace W spanned by the vectors $\mathbf{w}_j$, what tests would you make to verify if this is true?*

Problem 8 *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

a. (7 points) *Show that the columns of A are mutually orthogonal.*

b. (6 points) *Compute the determinant of A.*

c. (7 points) *Compute* A^{-1}. *(The simplest way to do this is to use the result of part a.)*

Problem 9 a. (10 points) *Fill in the entries* a, b *in the matrix*

$$A = \begin{pmatrix} 2 & 6 \\ a & b \end{pmatrix}$$

so that A *has the eigenvectors* $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *and* $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b. (10 points) *Find a different matrix* B *with those same eigenvectors* $\mathbf{v}_1, \mathbf{v}_2$*, and with eigenvalues* $\lambda_1 = 1, \lambda_2 = 0$*. Compute* B^{10}.

Brief Solutions:

1a. dim =1. (basis $t^2 - 2t$)

1.b. dim =2

2a.

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

2b.

$$\mathbf{x} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

unique solution.

3a. $\lambda = 16, 1, 0, 0$

3b.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

3c.

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

3d. $-A^{\mathrm{T}} = (-Q)\Sigma^{\mathrm{T}} P^{\mathrm{T}}$

4a.

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & -13 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

4b.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ -2 \\ 5 \\ 4 \end{pmatrix},$$

4c.

$$\mathbf{z}_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

4d. 0

5a. $\lambda = 2, 4, -4$

5b.

$$S = \begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

6a.

$$K = \begin{pmatrix} -2 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

6b.

$$-4y_1^2 + (\frac{3+\sqrt{61}}{2})y_2^2 + (\frac{3-\sqrt{61}}{2})y_3^2$$

6c. indefinite

7a. If $A^{\mathrm{T}}\mathbf{y} = \mathbf{c}$ has exactly one solution then dim coker $(A) = 0$, so dim rng $A) = m$ and all m-tuples $\mathbf{b}$ are in the range. Thus $A\mathbf{x} = \mathbf{b}$ must have a solution. This is a contradiction!

7b. Check that the set $\{\mathbf{p}, \mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is linearly dependent and $\mathbf{b}-\mathbf{p} \perp \mathbf{w}_i$ for $i = 1, \cdots, n$.

8a. Easy verification. Also follows from the easily checked result $A^{\mathrm{T}}A = 2I_4$.

8b. det $(A) = 4$

8c. Since $A^{\mathrm{T}}A = 2I_4$ we have $A^{-1} = \frac{1}{2}A^{\mathrm{T}}$.

9a. $a = -1, b = 7$.

9b.

$$B = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}$$

$B^{10} = B$