Name: ______________________________

Math 4242/4457 Sec. 10 practice Midterm Exam II

There are a total of 100 points on this 55 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and ONE 8.5×11 inch sheet of notes are allowed, but no books, other notes, cell phones or other electronic devices are allowed. Do all of your calculations on this test paper.

Problem	Score
1.	______
2.	______
3.	______
4.	______
5.	______
Total:	______

Problem 1 *(20 points) Is each of these statements true or false? Give a brief justification of your answer, or a counterexample .*

(a) *True or false: If the kernel of an* $m \times n$ *matrix* A *is* $\{\mathbf{0}\}$ *then* A *is nonsingular.*

(b) *True or false: The* n*th Legendre polynomial is orthogonal to all polynomials of order* $> n$*, with respect to the inner product* $< p, q >= \int_{-1}^{1} p(t)q(t)\, dt$*.*

(c) *True or false: The quadratic form*

$$Q(\mathbf{v}) = \mathbf{v}^T K \mathbf{v}$$

where K *is positive definite, defines a linear function* $Q : R^n \longrightarrow R$*.*

(d)*True or false: A positive definite matrix is always invertible..*

Problem 2 *Let* $L : R^2 \to R^2$ *be the linear function such that*

$$L\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} 2x + y \\ -x \end{pmatrix}.$$

a. (5 points) What is the matrix A *expressing* L *in terms of the standard basis vectors* $\mathbf{e}_1, \mathbf{e}_2$*?*

b. (10 points) *Find the matrix* B *that represents this linear function with respect to the basis* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *where* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

c. (5 points) *What is the matrix expressing* L^{-1} *in terms of the standard basis vectors* $\mathbf{e}_1, \mathbf{e}_2$*?*

Problem 3 a. (10 points) *Apply the Gram-Schmidt process with the Euclidean inner product to construct an orthonormal basis for the subspace of* R^4 *with basis* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *where*

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

b. (10 points) *Repeat the Gram-Schmidt construction using the weighted inner product*

$$< \mathbf{u}, \mathbf{v} >= u_1 v_1 + 2u_2 v_2 + u_3 v_3 + u_4 v_4,$$

where $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$, *etc.*

Problem 4

$$A = \begin{pmatrix} 1 & 2 & 3 & h \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & k \end{pmatrix}.$$

a. (15 points) *Determine the dimensions of the 4 fundamental subspaces of A. Your answer will depend on the values of h and k. Classify all cases.*

b. (5 points) *For $h = k = 1$ give a basis for $rng(A)$. Is this a basis for R^3? Justify your answer.*

Problem 5 *Suppose we want to fit the data*

$$t \;=\; 0 \,|\, 1 \,|\, 2 \,|\, 3$$

$$y \;=\; 1 \,|\, 3 \,|\, 2 \,|\, 3.$$

with the line $y = \alpha + \beta t$ *where* α *and* β *are constants.*

(a) *(10 points) Write down the matrix equation* $A\mathbf{x} = \mathbf{y}$*, where* $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ *that would hold if the straight line fit all the data exactly.*

(b) *(5 points) Use least squares with the standard dot product to compute the* α, β *that give the best fit.*

(c)(5 points) This problem is equivalent to projection of the vector $\mathbf{y} = (1, 3, 2, 3)^T$ *on a certain subspace. Find a basis for that subspace and give the projection of* $\mathbf{y}$ *onto it.*

Solutions:

1a. F

1b. F

1c. F

1d. T

2a. $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

2b. $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

2c. $A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

3a.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

.

3b.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{15}} \begin{pmatrix} 0 \\ 1 \\ 3 \\ -2 \end{pmatrix}$$

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4a. Case 1: $k \neq h/2$. Then $r = 3$, dim $\ker(A) = 1$, dim $\operatorname{coker}(A) = 0$. Case 2: $k = h/2$. Then $r = 2$, dim $\ker(A) = 2$, dim $\operatorname{coker}(A) = 1$.

4b. Basis

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

also a basis for R^3.

5a.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix}.$$

5b.

$$y = \frac{3}{2} + \frac{1}{2}t$$

5c. Projection is

$$\begin{pmatrix} 3/2 \\ 2 \\ 5/2 \\ 3 \end{pmatrix}.$$