12-12-07
Math 4242

## Review:

Final Exam: 10:30-12:30, ME 108
Monday, Dec. 17

9 problems, usual rules, last page for comp

1) T-F, basic facts. dets, similar matrices, symm. matrices, orthog. –, eigenvalues, eigenvectors; char eqn, char poly., trace

2) ~~Linear transformations~~ / ops. $L: V \to V$
Computation of the matrix $A$ of $L$ w.r.t. a basis for $V$.

3) G-S (n-tuples, fcn. spaces)

4) ~~Gaussian elim.~~

5) orthog. projections

6) least squares, data fitting (not just lines)

$A\underset{\sim}{x} = \underset{\sim}{f}$

$A^T A \underset{\sim}{x}^* = A^T \underset{\sim}{f}$

7) $A_{m \times n}$ 4 subspaces & their relations

8) diag. of real symmetric matrices by orthog. matrices

9) singular value decomp.

Ex: $T: P_2 \to P_2$ $\quad P_2 = \{ a + bt^2 + c(t+1) \}$

basis $\{ t^2, t, 1 \}$, with $t^2 = v_3$, $t = v_2$, $1 = v_1$

$$T = (t+1)\frac{d}{dt} + 2$$

i.e. $T\,p(t) = (t+1)\frac{dp}{dt} + 2\,p(t)$

lin. op.

compute matrix $A$ of $T$ w.r.t. $\{ v_1, v_2, v_3 \}$, where $v_1 = 1$, $v_2 = t$, $v_3 = t^2$

$T[v_1] = a_{11} v_1 + a_{21} v_2 + a_{31} v_3$

$T[v_1] = (t+1)\frac{d1}{dt} + 2(1) = 2 = 2v_1 + 0v_2 + 0v_3$

$T[v_2] = (t+1)\frac{d}{dt}t + 2t = 3t + 1$

$= 1v_1 + 3v_2 + 0v_3$

$T[v_3] = (t+1)\frac{dt^2}{dt} + 2t^2 = 4t^2 + 2t$

$= 0v_1 + 2v_2 + 4v_3$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

eigenvalues: $\lambda = 2, 3, 4$

$V_2$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y = 0, z = 0, x = \alpha$ $\quad V_2 = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$V_3 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$z = 0, \; x = y$

$$V_3 = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad V_4 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x = \frac{y}{2}, \; z = \frac{y}{2}$

$$V_4 = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\underset{\sim}{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sim 1, \quad \underset{\sim}{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \sim 1 + t$$

$$\underset{\sim}{w}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \sim 1 + 2t + t^2$$

Check: $T\underset{\sim}{w}_3 = (t+1)\frac{d}{dt}(1 + 2t + t^2) + 2(1 + 2t + t^2)$

$= (t+1)(2t) + 2(1+t^2) + 4t + 2t + 2$

$= 4t^2 + 8t + 4 = 4(1 + 2t + t^2)$

$= 4\underset{\sim}{w}_3$

$T\underset{\sim}{w}_2 = (t+1)\frac{d}{dt}(1+t) + 2(1+t) = 3(1+t)$

$= 3\underset{\sim}{w}_2$

$L: R^2 \to R^3$, $M: R^3 \to R^2$

$L\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $M\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $M\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$M\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

find matrix A of L w.r.t. $\{e_1, e_2\}$ $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ basis

same for M matrix B

$L\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2}L\begin{pmatrix}1\\3\end{pmatrix} + \frac{1}{2}L\begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}1\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\3\end{pmatrix} + \beta\begin{pmatrix}1\\1\end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\alpha + \beta = 1$

$3\alpha + \beta = 0 \Rightarrow \beta = -3\alpha$

$= \begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix} = \frac{1}{2}\hat{e}_1 + 0\hat{e}_2 + \frac{1}{2}\hat{e}_3$ $\Rightarrow -2\alpha = 1$ $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$

$L\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$= \begin{pmatrix}-\frac{1}{2}\\0\\-\frac{1}{2}\end{pmatrix} = -\frac{1}{2}\hat{e}_1 + 0\hat{e}_2 - \frac{1}{2}\hat{e}_3$

$\underset{3\times 2}{A} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\0 & 0\\\frac{1}{2} & -\frac{1}{2}\end{pmatrix}$, $\underset{2\times 3}{B} = \begin{pmatrix}1 & 0 & -1\\-2 & 1 & 1\end{pmatrix}$

$M\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix} = e_1 - 2e_2$

$M\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = e_2$ $= -e_1 + e_2$

$M\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$P_2$, $\{1, t, t^2\}$

inner product $\langle p, q \rangle = \int_0^1 p(t)\, q(t)\, t^2\, dt$

~~find ON basis with Gram-Schmidt~~

$w_1 = 1$, $\langle 1, 1 \rangle = \int_0^1 t^2\, dt = \frac{1}{3} = \|1\|^2$

$\Rightarrow$ $\boxed{u_1 = \sqrt{3}}$

$w_2 = t - \langle t, \sqrt{3} \rangle \sqrt{3} = t - \frac{3}{4}$

$\int_0^1 t \sqrt{3}\, t^2\, dt = \frac{\sqrt{3}}{4}$

$\|t - \frac{3}{4}\|^2 = \int_0^1 (t - \frac{3}{4})^2\, t^2\, dt$

$= \int_0^1 (t^2 - \frac{3}{2}t + \frac{9}{16}) t^2\, dt = \int_0^1 (t^4 - \frac{3t^3}{2} + \frac{9}{16} t^2) dt$

$= \frac{1}{5} - \frac{3}{8} + \frac{3}{16} = \frac{1}{5} - \frac{3}{16} = \frac{16-15}{80}$

$= \frac{1}{80}$

$\therefore \|t - \frac{3}{4}\| = \frac{1}{\sqrt{80}}$

$u_2 = \sqrt{80}\,(t - \frac{3}{4})$

Project ~~3t+2 on~~