Name: ____________________

Math 4567. Final Exam (take home)

Due by December 23, 2009

There are a total of 180 points and 8 problems on this take home exam.

Problem	Score
1.	______
2.	______
3.	______
4.	______
5.	______
6.	______
7.	______
8.	______
Total:	______

1. (20 points) Chapter 6, page 168, Problem 8

   A semi-infinite string, with one end fixed at the origin, is stretched along the positive x-axis and released at rest from a position $y = f(x)$, $x \geq 0$. Derive the expression

   $$y(x,t) = \frac{2}{\pi} \int_o^\infty \cos(\alpha\ at) \sin \alpha x \int_0^\infty f(s) \sin \alpha s\ ds\ d\alpha.$$

   If $F(x)$, $-\infty < x < \infty$, is the odd extension of $f(x)$, show that this result reduces to the form

   $$y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)].$$

2. (15 points) Chapter 6, page 168, Problem 11

   Find the bounded harmonic function $u(x,y)$ in the semi-infinite strip $0 < x < 1$, $y > 0$, that satisfies the conditions

   $$u_y(x,0) = 0, \quad u(0,y) = 0, \quad u_x(1,y) = f(y).$$

   Show that the answer is:

   $$u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\sinh \alpha x \cos \alpha y}{\alpha \cosh \alpha} \int_0^\infty f(s) \cos \alpha s\ ds\ d\alpha.$$

3. (15 points) Chapter 6, page 173, Problem 2

   Derive the solution of the wave equation $y_{tt} = a^2 y_{xx}$, $(-\infty < x < \infty, t > 0)$, which satisfies the conditions $y(x,0) = f(x)$ and $y_t(x,0) = 0$ when $-\infty < x < \infty$:

   $$y(x,t) = \frac{1}{\pi} \int_0^\infty \cos(\alpha\ at) \int_{-\infty}^\infty f(s) \cos \alpha(s-x) ds\ d\alpha.$$

   Show that this solution can be written in the form

   $$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)].$$

4. (20 points) Find the eigenvalues and normalized eigenfunctions of the Sturm-Liouville system

   $$-x^2(x^2 y')' = \lambda y,\ y(1) = 0,\ y(2) = 0, \quad 1 \leq x \leq 2.$$

   What are the orthogonality relations for the eigenfunctions.

5. **a.** (15 points) Determine a formal eigenfunction series expansion for the solution $y(x)$ of

$$-y'' - \mu y = f(x), \ y'(0) = 0, \ y'(1) = 0, \quad 0 \le x \le 1,$$

where f is a given continuous function on $[0, 1]$.

b. (10 points) What happens if the parameter μ is an eigenvalue?

6. Laplace's equation in polar coordinates is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

a. (10 points) Use separation of variables to find the solution $u(r, \theta)$ of this equation **outside** the circle $r = a$ and satisfying the boundary condition

$$u(a, \theta) = f(\theta)$$

on the circle. Require that $u(r, \theta)$ is bounded and continuous for $r \ge a$. To make u single-valued, require that $u(r, \theta) = u(r, \theta + 2\pi)$. Here, $f(\theta)$ is a continuous function with sectionally continuous derivative such that $f(0) = f(2\pi)$.

b. (5 points) Show that formally the solution is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} \left(a_n \cos n\theta + b_n \sin n\theta \right), \tag{1}$$

and compute the coefficients a_n, b_n.

c. (5 points) Show that your formal solution is an actual solution of Laplace's equation satisfying the boundary conditions.

d. (15 points) By interchanging the order of summation and integration in (1), derive the Poisson integral formula for the solution:

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) \frac{1 - \rho^2}{[1 + \rho^2 - 2\rho \cos(\theta - \psi)]} d\psi,$$

where $\rho = a/r < 1$.

7. In the following two problems we use the following definition of the complex Fourier integral tranform and its inversion:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}d\lambda, \quad \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx.$$

Fourier transforms on $(-\infty,\infty)$ and Fourier series have interesting relations between them. Here is one. The periodization of a function f on $(-\infty,\infty)$ is defined as

$$P[f](x) = \sum_{m=-\infty}^{\infty} f(x + 2\pi m).$$

To guarantee convergence of the infinite sum we restrict ourselves to functions that decay rapidly at infinity. A useful space of such functions f is the *Schwartz class* of functions that are infinitely differentiable everywhere, and for which there exist constants $C_{n,q}$ (depending on f) such that $|x^n \frac{d^q}{dx^q} f| \le C_{n,q}$ for all x and for each $n, q = 0, 1, 2, \cdots$. (An example of such a function is $f(x) = e^{-x^2}$.)

a. (10 points) Show that if f is in the Schwartz class then its periodization has period 2π. (You can assume the true fact that $P[f](x)$ is continuous and continuously differentiable.)

b. (10 points) Expand $P[f](x)$ into a complex Fourier series

$$P[f](x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and show that the Fourier coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} P[f](t)e^{-int}dt$$

are given by

$$c_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-int}dt = \frac{1}{2\pi}\hat{f}(n)$$

where $\hat{f}(\lambda)$ is the complex Fourier transform of $f(x)$.

c. (5 points) Conclude that

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}, \tag{2}$$

so $P[f](x)$ tells us the value of $\hat{f}$ at the integer points $\lambda = n$, but not in general at the non-integer points. (For $x = 0$, equation (2) is known as the *Poisson summation formula.*)

8. Let $f(x) = \frac{a}{x^2+a^2}$ for $a > 0$.

a. (10 points) Show that $\hat{f}(\lambda) = \pi e^{-a|\lambda|}$. Hint: It is easier to work backwards.

b. (5 points) Use the Poisson summation formula to derive the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.$$

c. (10 points) What happens as $a \to 0+$? (Look at the $n = 0$ term on the left hand side.) Can you obtain the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from this?