Name: ____________________

Math 4567. Final Exam (take home)

Due by December 23, 2009

There are a total of 180 points and 8 problems on this take home exam.

Problem	Score
1.	______
2.	______
3.	______
4.	______
5.	______
6.	______
7.	______
8.	______
Total:	______

1. **(20 points) Chapter 6, page 168, Problem 8**

   A semi-infinite string, with one end fixed at the origin, is stretched along the positive x-axis and released at rest from a position $y = f(x)$, $x \geq 0$. Derive the expression

   $$y(x,t) = \frac{2}{\pi}\int_0^\infty \cos(\alpha\ at)\sin\alpha x \int_0^\infty f(s)\sin\alpha s\ dsd\alpha. \qquad (1)$$

   If $F(x)$, $-\infty < x < \infty$, is the odd extension of $f(x)$, show that this result reduces to the form

   $$y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)].$$

   Solution: The boundary value problem is

   $$\begin{array}{lll} 1) & y_{tt} - a^2 y_{xx} = 0, & x > 0, t > 0, \\ 2) & y(0,t) = 0, & t \geq 0, \\ 3) & y_t(x,0) = 0, & x \geq 0, \\ 4) & y(x,0) = f(x), & x \geq 0, \end{array}$$

   and $y(x,t)$ is bounded for all $x > 0$, $t > 0$.

   Using the Fourier method we write $y = X(x)T(t)$, substitute into the wave equation and get the Sturm-Liouville problem with boundary conditions:

   $$X'' + \lambda X = 0, \quad X(0) = 0, \quad |X| < B, \quad x > 0,$$

   and

   $$T'' + a^2\lambda T = 0, \quad T'(0) = 0 \quad |T(t)| < B, \quad t > 0.$$

   Case 1: $\lambda = \alpha^2$, $\alpha > 0$. The differential equation and boundary condition give $X_\alpha(x) = \sin\alpha x$ and this is bounded. The T equation and boundary condition give $T_\alpha(t) = \cos(\alpha a t)$, which is bounded.

   Case 2: $\lambda = 0$. The differential equation and boundary condition give $X(x) = x$ which isn't bounded. Thus 0 is not an eigenvalue.

Case 3: $\lambda = -\alpha^2$, $\alpha > 0$. The differential equation and boundary condition give $X_\alpha(x) = \sinh \alpha x$, but this is unbounded. Thus there are no negative generalized eigenvalues.

We look for a solution of the form

$$y(x,t) = \int_0^\infty B(\alpha) \sin \alpha x \cos(\alpha a t)\; d\alpha.$$

The initial condition

$$y(x,y) = f(x) = \int_0^\infty B(\alpha) \sin \alpha x\; d\alpha$$

implies from the Fourier sine transform that

$$B(\alpha) = \frac{2}{\pi} \int_0^\infty f(s) \sin \alpha s\; ds.$$

Substituting the expression for $B(\alpha)$ into the integral expansion for y gives the stated solution (1).

Using the identity

$$\sin \alpha x \cos(\alpha a t) = \frac{1}{2} \left(\sin \alpha(x + at) + \sin \alpha(x - at) \right),$$

and defining the function $F(u)$ by

$$F(u) = \int_0^\infty B(\alpha) \sin \alpha u\; d\alpha = \frac{2}{\pi} \int_0^\infty \sin \alpha u \int_0^\infty f(s) \sin \alpha s\; ds\; d\alpha,$$

we see that $F(u)$ is defined for all real u, $F(-u) = -F(u)$ and $F(x) = f(x)$ for $x > 0$, and from the identity we have

$$y(x,t) = \frac{1}{2} \left(F(x + at) + F(x - at) \right).$$

This is in accordance with the general solution of the wave equation.

2. **(15 points) Chapter 6, page 168, Problem 11**

Find the bounded harmonic function $u(x,y)$ in the semi-infinite strip $0 < x < 1$, $y > 0$, that satisfies the conditions

$$u_y(x,0) = 0, \quad u(0,y) = 0, \quad u_x(1,y) = f(y).$$

Show that the answer is

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh \alpha x \cos \alpha y}{\alpha \cosh \alpha}\int_0^\infty f(s)\cos \alpha s \; ds \; d\alpha.$$

Solution: Using the Fourier method we write $u = X(x)Y(y)$, substitute into the Laplace equation and get the Sturm-Liouville problem with boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = 0, \qquad 0 < x < 1,$$

and

$$Y'' - \lambda Y = 0, \quad Y'(0) = 0 \quad |Y(y)| < B, \quad y > 0.$$

Case 1: $\lambda = \alpha^2,\ \alpha > 0$. The differential equation and boundary condition give $X_\alpha(x) = \sin \alpha x$. The Y equation and boundary condition give $Y_\alpha(y) = \cosh(\alpha y)$, which is unbounded. Therefore there are no such eigenvalues.

Case 2: $\lambda = 0$. The differential equation and boundary condition give $X(x) = x$ which isn't bounded. Thus 0 is not an eigenvalue.

Case 3: $\lambda = -\alpha^2,\ \alpha > 0$. The differential equation and boundary condition give $X_\alpha(x) = \sinh \alpha x$. The corresponding Y equation and boundary condition gives $Y_\alpha(y) = \cos \alpha y$ which is bounded.

Thus we look for a solution of the form

$$u(x,y) = \int_0^\infty A(\alpha)\sinh \alpha x \cos(\alpha y) \; d\alpha.$$

The nonhomogeneous boundary condition

$$u_x(1,y) = f(y) = \int_0^\infty A(\alpha)\alpha \cosh \alpha \cos \alpha y \; d\alpha$$

implies from the Fourier cosine transform that

$$A(\alpha)\alpha \cosh \alpha = \frac{2}{\pi}\int_0^\infty f(s)\cos \alpha s \; ds.$$

Substituting the expression for $A(\alpha)$ into the integral expansion for u gives the stated solution.

3. **(15 points) Chapter 6, page 173, Problem 2**

Derive the solution of the wave equation $y_{tt} = a^2 y_{xx}$, $(-\infty < x < \infty, t > 0)$, which satisfies the conditions $y(x,0) = f(x)$ and $y_t(x,0) = 0$ when $-\infty < x < \infty$:

$$y(x,t) = \frac{1}{\pi}\int_0^\infty \cos(\alpha\ at)\int_{-\infty}^\infty f(s)\cos\alpha(s-x)ds\ d\alpha.$$

Show that this solution can be written in the form

$$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)].$$

Solution: The boundary value problem is

$$\begin{array}{lll} 1) & y_{tt} - a^2 y_{xx} = 0, & -\infty < x < \infty,\ t > 0, \\ 2) & y_t(x,0) = 0, & x \geq 0, \\ 4) & y(x,0) = f(x), & -\infty < x < \infty, \end{array}$$

and $y(x,t)$ is bounded for all x and $t > 0$.

Using the Fourier method we write $y = X(x)T(t)$, substitute into the wave equation and get the Sturm-Liouville problem with boundary conditions:

$$T'' + a^2\lambda T = 0, \quad T'(0) = 0 \quad |T(t)| < B, \quad t > 0.$$

$$X'' + \lambda X = 0, \quad |X| < B, \quad -\infty < x < \infty.$$

Case 1: $\lambda = \alpha^2$, $\alpha > 0$. The differential equation and boundary condition for T give $T_\alpha(t) = \cos\alpha at$ and this is bounded. The X equation gives $X_\alpha(x) = A\cos(\alpha x) + B\sin\alpha x$, which is bounded.

Case 2: $\lambda = 0$. The differential equation and boundary condition give $T(t) = 1$. The X -equation and boundedness give $X = 1$ Thus 0 is a generalized eigenvalue.

Case 3: $\lambda = -\alpha^2$, $\alpha > 0$. The differential equation and boundary condition give $T_\alpha(t) = \cosh\alpha at$, but this is unbounded. Thus there are no negative generalized eigenvalues.

Thus, we look for a solution of the form

$$y(x,t) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\cos(\alpha a t)\ d\alpha. \tag{2}$$

The initial condition

$$y(x,0) = f(x) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x] d\alpha \tag{3}$$

implies from the Fourier transform theorem that

$$A(\alpha) = \frac{1}{\pi}\int_0^\infty f(s)\cos\alpha s\ ds, \quad B(\alpha) = \frac{1}{\pi}\int_0^\infty f(s)\sin\alpha s\ ds$$

Substituting the expressions for $A(\alpha)$, $B(\alpha)$ into the integral expansion for y and the identity

$$\cos\alpha s\cos\alpha x + \sin\alpha s\sin\alpha x = \cos\alpha(s-x)$$

give the stated solution.

Using the identities

$$\sin\alpha x\cos(\alpha a t) = \frac{1}{2}\left(\sin\alpha(x+at) + \sin\alpha(x-at)\right),$$

$$\cos\alpha x\cos(\alpha a t) = \frac{1}{2}\left(\cos\alpha(x+at) + \cos\alpha(x-at)\right),$$

and recalling from (3) that

$$f(u) = \int_0^\infty [A(\alpha)\cos\alpha u + B(\alpha)\sin\alpha u] d\alpha$$

for all u, we see that (2) can be written as

$$y(x,t) = \frac{1}{2}\left(f(x+at) + f(x-at)\right).$$

This is in accordance with the general solution of the wave equation.

4. **(20 points)** Find the eigenvalues and normalized eigenfunctions of the Sturm-Liouville system

$$-x^2(x^2y')' = \lambda y,\ y(1)=0,\ y(2)=0, \quad 1 \le x \le 2.$$

What are the orthogonality relations for the eigenfunctions?

Solution: Here, $p(x) = \frac{1}{x^2}$, $r(x) = x^2$, $q(x) = 0$, $[a, b] = [1, 2]$, the boundary conditions are $a_1y(1) + a_2y'(1) = 0$, $b_1y(2) + b_2y'(2) = 0$ with $a - 1 = b_1 = 1$, $a_2 + b - 2 = 0$, and the inner product is

$$(f, g)_p = \int_1^2 f(x)g(x)\frac{dx}{x^2}.$$

The operator L is

$$L = -x^4\frac{d^2}{dx^2} - 2x^3\frac{d}{dx},$$

and the eigenvalue equation is

$$LX = \lambda X.$$

From the identity relating eigenvalues, eigenfunctions and boundary conditions in this case we have

$$\lambda(X, X)_p = \int_1^2 x^2(X')^2 dx \geq 0,$$

so that there are no negative eigenvalues. Further, if 0 were an eigenvalue then necessarily X would be a constant, vanishing at $x = 1$ and $x = 2$, hence $X \equiv 0$. Thus the only eigenvalues are positive.

To Solve the eigenvalue problem we make the change of variable $u = 1 - 1/x$, (the leading constant 1 is chosen only for convenience). In this variable we find $L = -\frac{d^2}{du^2}$, so the eigenfunctions in the case $\lambda = \alpha^2,\ \alpha > 0$ must take the form

$$X = A\cos\alpha u + B\sin\alpha u.$$

where X must vanish when $x = 1$, (i.e. $u = 0$) which implies $A = 0$. Thus we have $X = \sin\alpha u$ and this must vanish when $x = 2$ (i.e., $u = 1/2$). The determining equation for the eigenvalues is thus $\sin\frac{\alpha}{2} = 0$, so

$$\alpha_n = 2\pi n, \quad n = 1, 2, \cdots.$$

The corresponding eigenfunctions can be chosen as

$$X_n(x) = -\sin 2\pi nu = -\sin(2\pi n[1 - \frac{1}{x}]) = \sin\frac{2\pi n}{x}.$$

Since

$$(X_n, X_n)_p = \int_1^2 \sin^2(\frac{2\pi n}{x})\frac{dx}{x^2} = \frac{1}{2}\int_1^2 [1-\cos(\frac{4\pi n}{x})]\frac{dx}{x^2} = \frac{1}{4},$$

the ON basis of eigenfunctions is $\{\phi_n(x)\}$ where

$$\phi_n(x) = 2\sin(\frac{2\pi n}{x}), \quad n = 1, 2, \cdots.$$

We conclude that

$$(\phi_n, \phi_m)_p = \delta_{nm}.$$

5. **a.** (15 points) Determine a formal eigenfunction series expansion for the solution $y(x)$ of

$$-y'' - \mu y = f(x), \; y'(0) = 0, \; y'(1) = 0, \quad 0 \le x \le 1,$$

where f is a given continuous function on $[0, 1]$.

b. (10 points) What happens if the parameter μ is an eigenvalue?

Solution: With $L = -\frac{d^2}{dx^2}$, the Sturm-Liouville eigenvalue problem here is

$$LX = \lambda X, \quad 0 < x < 1, \quad X'(0) = 0, \; X'(1) = 0.$$

We have solved this problem several times before. The eigenvalues are $\lambda_n = n^2\pi^2$, $n = 1, 2, \cdots$, and $\lambda_0 = 0$ with an ON basis of eigenfunctions

$$\phi_0(x) = 1, \quad \phi_n(x) = \sqrt{2}\cos(n\pi x).$$

The inner product is

$$(g_1, g_2) = \int_0^1 g_1(x)g_2(x)dx.$$

We expand both f and y in terms of the basis:

$$y(x) = \sum_{j=0}^{\infty} c_j\phi_j(x), \quad f(x) = \sum_{j=0}^{\infty} b_j\phi_j(x), \qquad c_j = (y, \phi_j), \; b_j = (f, \phi_j).$$

The equation to be solved is $Ly - \mu y = f$. Taking the inner product of both sides of this equation with ϕ_j we find

$$(Ly - \mu y, \phi_j) = (f, \phi_j) = b_j.$$

Note that

$$(Ly - \mu y, \phi_j) = (Ly, \phi_j) - \mu(y, \phi_j) = (y, L\phi_j) - \mu c_j = (\lambda_j - \mu)c_j.$$

Thus $(j^2\pi^2 - \mu)c_j = b_j$ and

$$c_j = \frac{b_j}{j^2\pi^2 - \mu}, \quad j = 0, 1, \cdots,$$

as long as $\mu \neq j^2\pi^2$ for some j. The solution is then

$$y(x) = \sum_{j=0}^{\infty} \frac{(f, \phi_j)}{j^2\pi^2 - \mu}\phi_j(x) = 2\sum_{j=0}^{\infty} \frac{\int_0^1 f(s)\cos j\pi s \; ds}{j^2\pi^2 - \mu}\cos j\pi x.$$

If $\mu = N^2\pi^2$ for some integer N, however, there is no longer a unique solution and maybe no solution at all. Adding any multiple $K\phi_N(x)$ to y will not change $(L-\mu)y$, because $(L-\mu)\phi_n = 0$. If there is a solution y to $(L - N^2\pi^2)y = f$ with y satisfying the boundary conditions then

$$(f, \phi_N) = ((L - N^2\pi^2)y, \phi_N) = (y, (L - N^2\pi^2)\phi_N) = 0,$$

so f is orthogonal to ϕ_N. In this case there is a solution but it is not unique. However, if f is not orthogonal to ϕ_N there is no solution.

6. Laplace's equation in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

a. (10 points) Use separation of variables to find the solution $u(r, \theta)$ of this equation **outside** the circle $r = a$ and satisfying the boundary condition

$$u(a, \theta) = f(\theta)$$

on the circle. Require that $u(r, \theta)$ is bounded and continuous for $r \geq a$. To make u single-valued, require that $u(r, \theta) = u(r, \theta + 2\pi)$.

Here, $f(\theta)$ is a continuous function with sectionally continuous derivative such that $f(0) = f(2\pi)$.

Solution:
Write $u = \Theta(\theta)R(r)$ and separate variables to derive the Sturm-Liouville eigenvalue problem with periodic boundary conditions

$$\Theta'' + \lambda\Theta = 0,\ -\pi < \theta < \pi, \quad \Theta(-\pi) = \Theta(\pi),\ \Theta'(-\pi) = \Theta'(\pi).$$

The conditions on R are

$$r^2R'' + rR - \lambda R = 0,\ a < r < \infty,$$

and $|R(r)| < B$ for some constant B and all $r > a$. We already showed in class that the possible eigenvalues in this case are $\lambda_n = n^2$, $n = 1, 2, \cdots$ with multiplicity two and eigenfunctions $\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta$, and $\lambda_0 = 0$ with eigenfunction $\Theta_0(\theta) = 1$, For $\lambda_n = n^2 > 0$ the possible solutions for R are $R(r) = Ar^n + Br^{-n}$. The boundedness condition requires $R_n(r) = r^{-n}$. For $\lambda = 0$ the possible solutions for R are $R(r) = A + B\ln r$, and the boundedness condition requires $R_0(r) = 1$. Thus we have the general solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{(a_n \cos n\theta + b_n \sin n\theta)}{r^n}.$$

b. (5 points) Show that formally the solution is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right), \tag{4}$$

and compute the coefficients a_n, b_n.

Solution: The initial condition

$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

implies via Fourier series that

$$\frac{a_j}{a^j} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos j\psi\ d\psi, \quad j = 0, 1, 2\cdots,$$

$$\frac{b_n}{a^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin n\psi\ d\psi, \quad n = 1, 2\cdots.$$

c. (5 points) Show that your formal solution is an actual solution of Laplace's equation satisfying the boundary conditions.

Solution: For $r = a$ we know from the Fourier convergence theorem that the boundary conditions are satisfied and the series converges uniformly and absolutely on the boundary. For $r > a$ we see that the series is a power series in $\rho = a/r < 1$ which converges for $\rho| = 1$. Thus the radius of convergence of this power series in ρ must be at least 1, and the series defines an analytic function of ρ for all $\rho < 1$. In particular, term - by term differentiation of the series is valid and the formal solution is an actual solution of Laplace's equation.

d. (15 points) By interchanging the order of summation and integration in (4), derive the Poisson integral formula for the solution:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)\frac{1-\rho^2}{[1+\rho^2-2\rho\cos(\theta-\psi)]}d\psi,$$

where $\rho = a/r < 1$.

Solution: Substituting the integral expressions for a_j, b_n in (4) and interchanging the order of summation and integration we obtain

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{a^n}{r^n}(\cos n\theta\cos n\psi + \sin n\theta\sin n\psi)\right]d\psi.$$

Note that

$$\frac{1}{2} + \sum_{n=1}^{\infty}\frac{a^n}{r^n}(\cos n\theta\cos n\psi + \sin n\theta\sin n\psi) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{a^n}{r^n}\cos n(\theta-\psi)$$

$$= \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\frac{a^n}{r^n}\{\exp in(\theta-\psi) + \exp -in(\theta-\psi)\}\right]$$

$$= \frac{1}{2}\left[-1 + \sum_{j=0}^{\infty}[(\frac{a}{r}e^{i(\theta-\psi)})^j + (\frac{a}{r}e^{-i(\theta-\psi)})^j]\right]$$

$$= \frac{1}{2}\left[-1 + \frac{1}{1-\frac{a}{r}e^{i(\theta-\psi)}} + \frac{1}{1-\frac{a}{r}e^{-i(\theta-\psi)}}\right]$$

$$= \frac{1}{2}\left[\frac{-(1-\frac{a}{r}e^{i(\theta-\psi)})(1-\frac{a}{r}e^{-i(\theta-\psi)}) + (1-\frac{a}{r}e^{-i(\theta-\psi)}) + (1-\frac{a}{r}e^{-i(\theta-\psi)})}{(1-\frac{a}{r}e^{i(\theta-\psi)})(1-\frac{a}{r}e^{-i(\theta-\psi)})}\right]$$

$$= \frac{1}{2}\frac{1-\frac{a^2}{r^2}}{1+\frac{a^2}{r^2}-2\frac{a}{r}\cos(\theta-\psi)} = \frac{1}{2}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\psi)},$$

where $\rho = a/r$ and we have used the formula

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \quad \text{if } |z| < 1,$$

for the sum of a geometric series. Thus we conclude that

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)\frac{1-\rho^2}{[1+\rho^2-2\rho\cos(\theta-\psi)]}d\psi,$$

where $\rho = a/r < 1$.

7. Fourier transforms on $(-\infty,\infty)$ and Fourier series have interesting relations between them. The periodization of a function f on $(-\infty,\infty)$ is defined as

$$P[f](x) = \sum_{m=-\infty}^{\infty} f(x+2\pi m).$$

To guarantee convergence of the infinite sum we restrict ourselves to functions that decay rapidly at infinity. A useful space of such functions f is the *Schwartz class* of functions that are infinitely differentiable everywhere, and for which there exist constants $C_{n,q}$ (depending on f) such that $|x^n \frac{d^q}{dx^q} f| \le C_{n,q}$ for all x and for each $n, q = 0, 1, 2, \cdots$. (An example of such a function is $f(x) = e^{-x^2}$.)

a. (10 points) Show that if f is in the Schwartz class then its periodization has period 2π. (You can assume the true fact that $P[f](x)$ is continuous and continuously differentiable.)

Solution:

$$P[f](x+2\pi) = \sum_{m=-\infty}^{\infty} f(x+2\pi+2\pi m) = \sum_{m=-\infty}^{\infty} f(x+2\pi(m+1))$$

$$= \sum_{k=-\infty}^{\infty} f(x+2\pi k) = P[f](x),$$

where $k = m+1$.

b. (10 points) Expand $P[f](x)$ into a complex Fourier series

$$P[f](x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and show that the Fourier coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} P[f](t)e^{-int}dt$$

are given by

$$c_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-int}dt = \frac{1}{2\pi}\hat{f}(n)$$

where $\hat{f}(\lambda)$ is the complex Fourier transform of $f(x)$.

Solution:

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} P[f](t)e^{-int}dt = \frac{1}{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{\infty} f(t+2\pi m)dt$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} f(t+2\pi m)e^{-int}dt$$

where the interchange of summation and integration is justified by the uniform convergence of the infinite series. Then, making the change of variable $\tau = t + 2\pi m$ in the integrals and using the fact that $e^{2\pi mi} = 1$, we have

$$c_n = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{2\pi m}^{2\pi(m+1)} f(\tau)e^{-in\tau}e^{2\pi mi}d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-in\tau}d\tau = \frac{1}{2\pi}\hat{f}(n),$$

where $\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(\tau)e^{-i\lambda\tau}d\tau$.

c. (5 points) Conclude that

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx}, \tag{5}$$

so $P[f](x)$ tells us the value of $\hat{f}$ at the integer points $\lambda = n$, but not in general at the non-integer points. (For $x = 0$, equation (5) is known as the *Poisson summation formula.*)

Solution: We have

$$P[f](x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Substituting the definition of $P[f](x)$ and the computed values of c_n into this formula we obtain equation (5).

8. Let $f(x) = \frac{a}{x^2+a^2}$ for $a > 0$.

a. (10 points) Show that $\hat{f}(\lambda) = \pi e^{-a|\lambda|}$. Hint: It is easier to work backwards.

Solution: The Fourier transform pair is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda, \quad \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx,$$

and this transform is invertible. Staring with $\hat{f}(\lambda) = \pi e^{-|\lambda|}$ and making use of the indefinite integral

$$\int e^u \cos bu \; du = e^u \frac{\cos bu + b \sin bu}{1+b^2} + C,$$

we find

$$\begin{aligned} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-a|\lambda|} e^{i\lambda x} d\lambda \\ &= \int_0^{\infty} e^{-a\lambda} \cos \lambda x \; d\lambda \\ = -e^{-a\lambda} \frac{a \cos \lambda x - x \sin \lambda x}{a^2+x^2} \Big|_{\lambda=0}^{\lambda \to +\infty} &= \frac{a}{a^2+x^2}. \end{aligned}$$

b. (5 points) Use the Poisson summation formula to derive the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.$$

Solution: Substituting $f(x) = \frac{a}{a^2+x^2}$ and $\hat{f}(\lambda) = \pi e^{-|\lambda|}$ into the Poisson summation formula we find

$$\sum_{n=-\infty}^{\infty} \frac{a}{a^2+(x+2\pi n)^2} = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-a|n|} e^{inx} = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-an} \cos nx.$$

Now set $x = 0$ to get

$$\sum_{n=-\infty}^{\infty} \frac{a}{a^2 + (2\pi n)^2} = -\frac{1}{2} + \sum_{n=0}^{\infty} e^{-an} = -\frac{1}{2} + \frac{1}{1 - e^{-a}}.$$

Thus

$$\sum_{n=-\infty}^{\infty} \frac{a}{a^2 + (2\pi n)^2} = \frac{1 + e^{-a}}{2(1 - e^{-a})}.$$

Changing parameters to $a = 2\pi b$ gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{b^2 + n^2} = \frac{\pi}{b} \frac{1 + e^{-2\pi b}}{(1 - e^{-2\pi b})}.,$$

which, modulo setting $b = a$ is the desired result.

c. (10 points) What happens as $a \to 0+$? (Look at the $n = 0$ term on the left hand side.) Can you obtain the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from this?

Solution: The $n = 0$ term on the right hand side gives us $\frac{1}{a^2}$, so we can't immediately set $a = 0$. Instead we subtract $\frac{1}{a^2}$ from both sides of the equation, to get

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + a^2} = \frac{\pi}{a} \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} - \frac{1}{a^2} = \frac{(\pi a - 1) + (\pi a + 1)e^{-2\pi a}}{a^2(1 - e^{-2\pi a})}.$$

The limit of the left hand side as $a \to 0$ is obvious: $2\sum_{n=1}^{\infty} \frac{1}{n^2}$. To get the limit on the right hand side we expand the numerator and the denominator in a power series in a. The leading term in the numerator is $\frac{2}{3}\pi^3 a^3 + \cdots$, in the denominator it is $2\pi a^3 + \cdots$. Thus the limit is $\pi^2/3$. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$